JOURNAL OF FORMALIZED MATHEMATICS
Volume 6, Released 1994, Published 2003
Inst. of Computer Science, Univ. of Białystok

Introduction to Circuits, I[1]

Yatsuka Nakamura
Shinshu University, Nagano

Piotr Rudnicki
University of Alberta, Edmonton

Andrzej Trybulec
Warsaw University, Białystok

Pauline N. Kawamoto
Shinshu University, Nagano

Summary. This article is the third in a series of four articles (preceded by [20],[21] and continued in [22]) about modelling circuits by many sorted algebras.

A circuit is defined as a locally-finite algebra over a circuit-like many sorted signature. For circuits we define notions of input function and of circuit state which are later used (see [22]) to define circuit computations. For circuits over monotonic signatures we introduce notions of vertex size and vertex depth that characterize certain graph properties of circuit's signature in terms of elements of its free envelope algebra. The depth of a finite circuit is defined as the maximal depth over its vertices.

MML Identifier: `CIRCUIT1`.

WWW: `http://mizar.org/JFM/Vol6/circuit1.html`

The articles [25], [15], [32], [5], [4], [29], [2], [28], [33], [14], [11], [34], [18], [30], [1], [23], [6], [31], [16], [7], [3], [8], [9], [10], [17], [12], [26], [27], [13], [19], [24], [20], and [21] provide the notation and terminology for this paper.

1. Circuit State

Let S be a non void circuit-like non empty many sorted signature. A circuit of S is a locally-finite algebra over S.

In the sequel I_1 is a circuit-like non void non empty many sorted signature.

Let us consider I_1 and let S_1 be a non-empty circuit of I_1. The functor Set-Constants(S_1) yielding a many sorted set indexed by SortsWithConstants(I_1) is defined as follows:

(Def. 1) For every vertex x of I_1 such that $x \in \operatorname{dom}\text{Set-Constants}(S_1)$ holds $(\text{Set-Constants}(S_1))(x) \in \text{Constants}(S_1, x)$.

One can prove the following proposition

(1) Let given I_1, S_1 be a non-empty circuit of I_1, v be a vertex of I_1, and e be an element of (the sorts of $S_1)(v)$. If $v \in \text{SortsWithConstants}(I_1)$ and $e \in \text{Constants}(S_1, v)$, then $(\text{Set-Constants}(S_1))(v) = e$.

Let us consider I_1 and let C_1 be a circuit of I_1. An input function of C_1 is a many sorted function from $\text{InputVertices}(I_1) \longmapsto \mathbb{N}$ into (the sorts of $C_1) \restriction \text{InputVertices}(I_1)$.

One can prove the following proposition

[1]Partial funding for this work has been provided by: Shinshu Endowment Fund for Information Science, NSERC Grant OGP9207, JSTF award 651-93-S009.

(2) Let given I_1, S_1 be a non-empty circuit of I_1, I_2 be an input function of S_1, and n be a natural number. If I_1 has input vertices, then $(\text{commute}(I_2))(n)$ is an input assignment of S_1.

Let us consider I_1. Let us assume that I_1 has input vertices. Let S_1 be a non-empty circuit of I_1, let I_2 be an input function of S_1, and let n be a natural number. The functor n-th-input(I_2) yielding an input assignment of S_1 is defined as follows:

(Def. 2) n-th-input$(I_2) = (\text{commute}(I_2))(n)$.

Let us consider I_1 and let S_1 be a circuit of I_1. A state of S_1 is an element of $\prod$(the sorts of S_1).
One can prove the following two propositions:

(4)[1] For every I_1 and for every non-empty circuit S_1 of I_1 and for every state s of S_1 holds $\text{dom}\, s =$ the carrier of I_1.

(5) Let given I_1, S_1 be a non-empty circuit of I_1, s be a state of S_1, and v be a vertex of I_1. Then $s(v) \in (\text{the sorts of } S_1)(v)$.

Let us consider I_1, let S_1 be a non-empty circuit of I_1, let s be a state of S_1, and let o be an operation symbol of I_1. The functor o depends-on-in s yields an element of $\text{Args}(o, S_1)$ and is defined as follows:

(Def. 3) o depends-on-in $s = s \cdot \text{Arity}(o)$.

In the sequel I_1 is a monotonic circuit-like non void non empty many sorted signature.
Next we state the proposition

(6) Let given I_1, S_1 be a locally-finite non-empty algebra over I_1, v, w be vertices of I_1, e_1 be an element of (the sorts of $\text{FreeEnvelope}(S_1))(v)$, and q_1 be a decorated tree yielding finite sequence. Suppose $v \in \text{InnerVertices}(I_1)$ and $e_1 = \langle$the action at v, the carrier of $I_1\rangle$-tree(q_1). Let k be a natural number. If $k \in \text{dom}\, q_1$ and $q_1(k) \in (\text{the sorts of FreeEnvelope}(S_1))(w)$, then $w = \text{Arity}(\text{the action at } v)_k$.

Let us consider I_1, let S_1 be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1. Note that every element of (the sorts of $\text{FreeEnvelope}(S_1))(v)$ is finite, non empty, function-like, and relation-like.

Let us consider I_1, let S_1 be a locally-finite non-empty algebra over I_1, and let v be a vertex of I_1. Note that every element of (the sorts of $\text{FreeEnvelope}(S_1))(v)$ is decorated tree-like.

The following four propositions are true:

(7) Let given I_1, S_1 be a locally-finite non-empty algebra over I_1, v, w be vertices of I_1, e_1 be an element of (the sorts of $\text{FreeEnvelope}(S_1))(v)$, e_2 be an element of (the sorts of $\text{FreeEnvelope}(S_1))(w)$, q_1 be a decorated tree yielding finite sequence, and k_1 be a natural number. Suppose $v \in \text{InnerVertices}(I_1) \setminus \text{SortsWithConstants}(I_1)$ and $e_1 = \langle$the action at v, the carrier of $I_1\rangle$-tree(q_1) and $k_1 + 1 \in \text{dom}\, q_1$ and $q_1(k_1+1) \in (\text{the sorts of FreeEnvelope}(S_1))(w)$. Then e_1 with-replacement$(\langle k_1\rangle, e_2) \in$ (the sorts of $\text{FreeEnvelope}(S_1))(v)$.

(8) Let given I_1, A be a locally-finite non-empty algebra over I_1, v be an element of I_1, and e be an element of (the sorts of $\text{FreeEnvelope}(A))(v)$. Suppose $1 < \text{card}\, e$. Then there exists an operation symbol o of I_1 such that $e(\emptyset) = \langle o, \text{the carrier of } I_1\rangle$.

(9) Let I_1 be a non void circuit-like non empty many sorted signature, S_1 be a non-empty circuit of I_1, s be a state of S_1, and o be an operation symbol of I_1. Then $(\text{Den}(o, S_1))(o$ depends-on-in $s) \in (\text{the sorts of } S_1)(\text{the result sort of } o)$.

(10) Let given I_1, A be a non-empty circuit of I_1, v be a vertex of I_1, and e be an element of (the sorts of $\text{FreeEnvelope}(A))(v)$. Suppose $e(\emptyset) = \langle$the action at v, the carrier of $I_1\rangle$. Then there exists a decorated tree yielding finite sequence p such that $e = \langle$the action at v, the carrier of $I_1\rangle$-tree(p).

[1] The proposition (3) has been removed.

2. Vertex Size

Let I_1 be a monotonic non void non empty many sorted signature, let A be a locally-finite non-empty algebra over I_1, and let v be a sort symbol of I_1. Note that (the sorts of FreeEnvelope$(A))(v)$ is finite.

Let us consider I_1, let A be a locally-finite non-empty algebra over I_1, and let v be a sort symbol of I_1. The functor size(v,A) yields a natural number and is defined by:

(Def. 4) There exists a finite non empty subset s of $\mathbb{N}$ such that $s = \{\mathrm{card}\, t : t$ ranges over elements of (the sorts of FreeEnvelope$(A))(v)\}$ and size$(v,A) = \max s$.

One can prove the following propositions:

(11) Let given I_1, A be a locally-finite non-empty algebra over I_1, and v be an element of I_1. Then size$(v,A) = 1$ if and only if $v \in$ InputVertices$(I_1) \cup$ SortsWithConstants(I_1).

(12) Let given I_1, S_1 be a locally-finite non-empty algebra over I_1, v, w be vertices of I_1, e_1 be an element of (the sorts of FreeEnvelope$(S_1))(v)$, e_2 be an element of (the sorts of FreeEnvelope$(S_1))(w)$, and q_1 be a decorated tree yielding finite sequence. Suppose $v \in$ InnerVertices$(I_1) \setminus$ SortsWithConstants(I_1) and card $e_1 =$ size(v,S_1) and $e_1 = \langle$the action at v, the carrier of $I_1\rangle$-tree(q_1) and $e_2 \in \mathrm{rng}\, q_1$. Then card $e_2 =$ size(w,S_1).

(13) Let given I_1, A be a locally-finite non-empty algebra over I_1, v be a vertex of I_1, and e be an element of (the sorts of FreeEnvelope$(A))(v)$. Suppose $v \in$ InnerVertices$(I_1) \setminus$ SortsWithConstants(I_1) and card $e =$ size(v,A). Then there exists a decorated tree yielding finite sequence q such that $e = \langle$the action at v, the carrier of $I_1\rangle$-tree(q).

(14) Let given I_1, A be a locally-finite non-empty algebra over I_1, v be a vertex of I_1, and e be an element of (the sorts of FreeEnvelope$(A))(v)$. Suppose $v \in$ InnerVertices$(I_1) \setminus$ SortsWithConstants(I_1) and card $e =$ size(v,A). Then there exists an operation symbol o of I_1 such that $e(\emptyset) = \langle o$, the carrier of $I_1\rangle$.

Let S be a non void non empty many sorted signature, let A be a locally-finite non-empty algebra over S, let v be a sort symbol of S, and let e be an element of (the sorts of FreeEnvelope$(A))(v)$. The functor depth(e) yielding a natural number is defined as follows:

(Def. 5) There exists an element e' of (the sorts of Free(the sorts of $A))(v)$ such that $e = e'$ and depth$(e) =$ depth(e').

The following propositions are true:

(15) Let given I_1, A be a locally-finite non-empty algebra over I_1, and v, w be elements of I_1. If $v \in$ InnerVertices(I_1) and $w \in \mathrm{rng}\,$Arity(the action at v), then size$(w,A) <$ size(v,A).

(16) For every I_1 and for every locally-finite non-empty algebra A over I_1 and for every sort symbol v of I_1 holds size$(v,A) > 0$.

(17) Let given I_1, A be a non-empty circuit of I_1, v be a vertex of I_1, e be an element of (the sorts of FreeEnvelope$(A))(v)$, and p be a decorated tree yielding finite sequence. Suppose that

(i) $v \in$ InnerVertices(I_1),

(ii) $e = \langle$the action at v, the carrier of $I_1\rangle$-tree(p), and

(iii) for every natural number k such that $k \in \mathrm{dom}\, p$ there exists an element e_3 of (the sorts of FreeEnvelope$(A))($Arity(the action at $v)_k)$ such that $e_3 = p(k)$ and card $e_3 =$ size(Arity(the action at $v)_k, A)$.

Then card $e =$ size(v,A).

3. Vertex and Circuit Depth

Let S be a monotonic non void non empty many sorted signature, let A be a locally-finite non-empty algebra over S, and let v be a sort symbol of S. The functor $\mathrm{depth}(v,A)$ yields a natural number and is defined by:

(Def. 6) There exists a finite non empty subset s of $\mathbb{N}$ such that $s = \{\mathrm{depth}(t) : t$ ranges over elements of (the sorts of $\mathrm{FreeEnvelope}(A))(v)\}$ and $\mathrm{depth}(v,A) = \max s$.

Let I_1 be a finite monotonic circuit-like non void non empty many sorted signature and let A be a non-empty circuit of I_1. The functor $\mathrm{depth}(A)$ yielding a natural number is defined as follows:

(Def. 7) There exists a finite non empty subset D_1 of $\mathbb{N}$ such that $D_1 = \{\mathrm{depth}(v,A); v$ ranges over elements of I_1: $v \in$ the carrier of $I_1\}$ and $\mathrm{depth}(A) = \max D_1$.

We now state three propositions:

(18) Let I_1 be a finite monotonic circuit-like non void non empty many sorted signature, A be a non-empty circuit of I_1, and v be a vertex of I_1. Then $\mathrm{depth}(v,A) \leq \mathrm{depth}(A)$.

(19) Let given I_1, A be a non-empty circuit of I_1, and v be a vertex of I_1. Then $\mathrm{depth}(v,A) = 0$ if and only if $v \in \mathrm{InputVertices}(I_1)$ or $v \in \mathrm{SortsWithConstants}(I_1)$.

(20) Let given I_1, A be a locally-finite non-empty algebra over I_1, and v, v_1 be sort symbols of I_1. If $v \in \mathrm{InnerVertices}(I_1)$ and $v_1 \in \mathrm{rng}\,\mathrm{Arity}(\text{the action at } v)$, then $\mathrm{depth}(v_1,A) < \mathrm{depth}(v,A)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek. Introduction to trees. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/trees_1.html`.

[4] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[5] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal2.html`.

[6] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[7] Grzegorz Bancerek. Cartesian product of functions. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/funct_6.html`.

[8] Grzegorz Bancerek. König's Lemma. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/trees_2.html`.

[9] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/trees_3.html`.

[10] Grzegorz Bancerek. Joining of decorated trees. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/trees_4.html`.

[11] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[12] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/dtconstr.html`.

[13] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/msualg_2.html`.

[14] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[15] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[16] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_4.html`.

[17] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar — part I. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lang1.html.

[18] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[19] Beata Madras. Products of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pralg_2.html.

[20] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pre_circ.html.

[21] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, II. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree2.html.

[22] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, II. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/circuit2.html.

[23] Andrzej Nędzusiak. σ-fields and probability. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/prob_1.html.

[24] Beata Perkowska. Free many sorted universal algebra. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree.html.

[25] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[26] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[27] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[28] Andrzej Trybulec. On the sets inhabited by numbers. *Journal of Formalized Mathematics*, 15, 2003. http://mizar.org/JFM/Vol16/membered.html.

[29] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[30] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/group_1.html.

[31] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[32] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[33] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[34] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received December 15, 1994

Published January 2, 2004