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Domains and Their Cartesian Products

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Summary. The article includes: theorems related to domains, theorems related to Cartesian products presented earlier in various articles and simplified here by substituting domains for sets and omitting the assumption that the sets involved must not be empty. Several schemes and theorems related to Fraenkel operator are given. We also redefine subset yielding functions such as the pair of elements of a set and the union of two subsets of a set.

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The articles [3], [2], [1], [5], and [4] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: a, b, c, d denote sets, D, X_1, X_2, X_3, X_4 denote non empty sets, x_1, y_1, z_1 denote elements of X_1, x_2 denotes an element of X_2, x_3 denotes an element of X_3, x_4 denotes an element of X_4, and A_1, B_1 denote subsets of X_1.

The following propositions are true:

(9)[1] If $a \in [: X_1, X_2 :]$, then there exist x_1, x_2 such that $a = \langle x_1, x_2 \rangle$.

(12)[2] For all elements x, y of $[: X_1, X_2 :]$ such that $x_{\mathbf{1}} = y_{\mathbf{1}}$ and $x_{\mathbf{2}} = y_{\mathbf{2}}$ holds $x = y$.

Let us consider X_1, X_2, x_1, x_2. Then $\langle x_1, x_2 \rangle$ is an element of $[: X_1, X_2 :]$.

Let us consider X_1, X_2 and let x be an element of $[: X_1, X_2 :]$. Then $x_{\mathbf{1}}$ is an element of X_1. Then $x_{\mathbf{2}}$ is an element of X_2.

We now state three propositions:

(15)[3] $a \in [: X_1, X_2, X_3 :]$ iff there exist x_1, x_2, x_3 such that $a = \langle x_1, x_2, x_3 \rangle$.

(16) If for every a holds $a \in D$ iff there exist x_1, x_2, x_3 such that $a = \langle x_1, x_2, x_3 \rangle$, then $D = [: X_1, X_2, X_3 :]$.

(17) $D = [: X_1, X_2, X_3 :]$ iff for every a holds $a \in D$ iff there exist x_1, x_2, x_3 such that $a = \langle x_1, x_2, x_3 \rangle$.

In the sequel x, y denote elements of $[: X_1, X_2, X_3 :]$.

Let us consider X_1, X_2, X_3, x_1, x_2, x_3. Then $\langle x_1, x_2, x_3 \rangle$ is an element of $[: X_1, X_2, X_3 :]$.

The following propositions are true:

(24)[4] $a = x_{\mathbf{1}}$ iff for all x_1, x_2, x_3 such that $x = \langle x_1, x_2, x_3 \rangle$ holds $a = x_1$.

[1] The propositions (1)–(8) have been removed.
[2] The propositions (10) and (11) have been removed.
[3] The propositions (13) and (14) have been removed.
[4] The propositions (18)–(23) have been removed.

(25) $b = x_{\mathbf{2}}$ iff for all x_1, x_2, x_3 such that $x = \langle x_1, x_2, x_3\rangle$ holds $b = x_2$.

(26) $c = x_{\mathbf{3}}$ iff for all x_1, x_2, x_3 such that $x = \langle x_1, x_2, x_3\rangle$ holds $c = x_3$.

(28)[5] If $x_{\mathbf{1}} = y_{\mathbf{1}}$ and $x_{\mathbf{2}} = y_{\mathbf{2}}$ and $x_{\mathbf{3}} = y_{\mathbf{3}}$, then $x = y$.

(31)[6] $a \in [:X_1, X_2, X_3, X_4:]$ iff there exist x_1, x_2, x_3, x_4 such that $a = \langle x_1, x_2, x_3, x_4\rangle$.

(32) If for every a holds $a \in D$ iff there exist x_1, x_2, x_3, x_4 such that $a = \langle x_1, x_2, x_3, x_4\rangle$, then $D = [:X_1, X_2, X_3, X_4:]$.

(33) $D = [:X_1, X_2, X_3, X_4:]$ iff for every a holds $a \in D$ iff there exist x_1, x_2, x_3, x_4 such that $a = \langle x_1, x_2, x_3, x_4\rangle$.

In the sequel x denotes an element of $[:X_1, X_2, X_3, X_4:]$.

Let us consider X_1, X_2, X_3, X_4, x_1, x_2, x_3, x_4. Then $\langle x_1, x_2, x_3, x_4\rangle$ is an element of $[:X_1, X_2, X_3, X_4:]$.

Next we state several propositions:

(40)[7] $a = x_{\mathbf{1}}$ iff for all x_1, x_2, x_3, x_4 such that $x = \langle x_1, x_2, x_3, x_4\rangle$ holds $a = x_1$.

(41) $b = x_{\mathbf{2}}$ iff for all x_1, x_2, x_3, x_4 such that $x = \langle x_1, x_2, x_3, x_4\rangle$ holds $b = x_2$.

(42) $c = x_{\mathbf{3}}$ iff for all x_1, x_2, x_3, x_4 such that $x = \langle x_1, x_2, x_3, x_4\rangle$ holds $c = x_3$.

(43) $d = x_{\mathbf{4}}$ iff for all x_1, x_2, x_3, x_4 such that $x = \langle x_1, x_2, x_3, x_4\rangle$ holds $d = x_4$.

(45)[8] For all elements x, y of $[:X_1, X_2, X_3, X_4:]$ such that $x_{\mathbf{1}} = y_{\mathbf{1}}$ and $x_{\mathbf{2}} = y_{\mathbf{2}}$ and $x_{\mathbf{3}} = y_{\mathbf{3}}$ and $x_{\mathbf{4}} = y_{\mathbf{4}}$ holds $x = y$.

In the sequel A_2 denotes a subset of X_2, A_3 denotes a subset of X_3, and A_4 denotes a subset of X_4.

In this article we present several logical schemes. The scheme *Fraenkel1* concerns a unary predicate $\mathcal{P}$, and states that:

For every X_1 holds $\{x_1 : \mathcal{P}[x_1]\}$ is a subset of X_1

for all values of the parameters.

The scheme *Fraenkel2* concerns a binary predicate $\mathcal{P}$, and states that:

For all X_1, X_2 holds $\{\langle x_1, x_2\rangle : \mathcal{P}[x_1, x_2]\}$ is a subset of $[:X_1, X_2:]$

for all values of the parameters.

The scheme *Fraenkel3* concerns a ternary predicate $\mathcal{P}$, and states that:

For all X_1, X_2, X_3 holds $\{\langle x_1, x_2, x_3\rangle : \mathcal{P}[x_1, x_2, x_3]\}$ is a subset of $[:X_1, X_2, X_3:]$

for all values of the parameters.

The scheme *Fraenkel4* concerns a 4-ary predicate $\mathcal{P}$, and states that:

For all X_1, X_2, X_3, X_4 holds $\{\langle x_1, x_2, x_3, x_4\rangle : \mathcal{P}[x_1, x_2, x_3, x_4]\}$ is a subset of $[:X_1, X_2, X_3, X_4:]$

for all values of the parameters.

The scheme *Fraenkel5* concerns two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

For every X_1 such that for every x_1 such that $\mathcal{P}[x_1]$ holds $\mathcal{Q}[x_1]$ holds $\{y_1 : \mathcal{P}[y_1]\} \subseteq \{z_1 : \mathcal{Q}[z_1]\}$

for all values of the parameters.

The scheme *Fraenkel6* concerns two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

For every X_1 such that for every x_1 holds $\mathcal{P}[x_1]$ iff $\mathcal{Q}[x_1]$ holds $\{y_1 : \mathcal{P}[y_1]\} = \{z_1 : \mathcal{Q}[z_1]\}$

for all values of the parameters.

The scheme *SubsetD* deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

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[5] The proposition (27) has been removed.

[6] The propositions (29) and (30) have been removed.

[7] The propositions (34)–(39) have been removed.

[8] The proposition (44) has been removed.

$\{d; d$ ranges over elements of $\mathcal{A} : \mathcal{P}[d]\}$ is a subset of $\mathcal{A}$
for all values of the parameters.

One can prove the following propositions:

(48)[9] $X_1 = \{x_1\}$.

(49) $[: X_1, X_2 :] = \{\langle x_1, x_2 \rangle\}$.

(50) $[: X_1, X_2, X_3 :] = \{\langle x_1, x_2, x_3 \rangle\}$.

(51) $[: X_1, X_2, X_3, X_4 :] = \{\langle x_1, x_2, x_3, x_4 \rangle\}$.

(52) $A_1 = \{x_1 : x_1 \in A_1\}$.

(53) $[: A_1, A_2 :] = \{\langle x_1, x_2 \rangle : x_1 \in A_1 \land x_2 \in A_2\}$.

(54) $[: A_1, A_2, A_3 :] = \{\langle x_1, x_2, x_3 \rangle : x_1 \in A_1 \land x_2 \in A_2 \land x_3 \in A_3\}$.

(55) $[: A_1, A_2, A_3, A_4 :] = \{\langle x_1, x_2, x_3, x_4 \rangle : x_1 \in A_1 \land x_2 \in A_2 \land x_3 \in A_3 \land x_4 \in A_4\}$.

(56) $\emptyset_{(X_1)} = \{x_1 : \mathit{contradiction}^{11}\}$.

(57) $A_1{}^{\rm c} = \{x_1 : x_1 \notin A_1\}$.

(58) $A_1 \cap B_1 = \{x_1 : x_1 \in A_1 \land x_1 \in B_1\}$.

(59) $A_1 \cup B_1 = \{x_1 : x_1 \in A_1 \lor x_1 \in B_1\}$.

(60) $A_1 \setminus B_1 = \{x_1 : x_1 \in A_1 \land x_1 \notin B_1\}$.

(61) $A_1 \dot{-} B_1 = \{x_1 : x_1 \in A_1 \land x_1 \notin B_1 \lor x_1 \notin A_1 \land x_1 \in B_1\}$.

(62) $A_1 \dot{-} B_1 = \{x_1 : x_1 \notin A_1 \Leftrightarrow x_1 \in B_1\}$.

(63) $A_1 \dot{-} B_1 = \{x_1 : x_1 \in A_1 \Leftrightarrow x_1 \notin B_1\}$.

(64) $A_1 \dot{-} B_1 = \{x_1 : x_1 \in A_1 \Leftrightarrow x_1 \notin B_1\}$.

Let D be a non empty set and let x_1 be an element of D. Then $\{x_1\}$ is a subset of D. Let x_2 be an element of D. Then $\{x_1, x_2\}$ is a subset of D. Let x_3 be an element of D. Then $\{x_1, x_2, x_3\}$ is a subset of D. Let x_4 be an element of D. Then $\{x_1, x_2, x_3, x_4\}$ is a subset of D. Let x_5 be an element of D. Then $\{x_1, x_2, x_3, x_4, x_5\}$ is a subset of D. Let x_6 be an element of D. Then $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a subset of D. Let x_7 be an element of D. Then $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ is a subset of D. Let x_8 be an element of D. Then $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ is a subset of D.

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[9] The propositions (46) and (47) have been removed.

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