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Axioms of Incidency

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Summary. This article is based on *"Foundations of Geometry"* by Karol Borsuk and Wanda Szmielew ([1]). The fourth axiom of incidency is weakened. In [1] it has the form *for any plane there exist three non-collinear points in the plane* and in the article *for any plane there exists one point in the plane*. The original axiom is proved. The article includes: theorems concerning collinearity of points and coplanarity of points and lines, basic theorems concerning lines and planes, fundamental existence theorems, theorems concerning intersection of lines and planes.

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The articles [5], [4], [2], [6], [3], and [7] provide the notation and terminology for this paper.

We consider projective incidence structures as systems

⟨ points, lines, an incidence ⟩,

where the points and the lines constitute non empty sets and the incidence is a relation between the points and the lines.

We introduce incidence structures which are extensions of projective incidence structure and are systems

⟨ points, lines, planes, an incidence, an incidence2, an incidence3 ⟩,

where the points, the lines, and the planes constitute non empty sets, the incidence is a relation between the points and the lines, the incidence2 is a relation between the points and the planes, and the incidence3 is a relation between the lines and the planes.

Let S be a projective incidence structure. A point of S is an element of the points of S. A line of S is an element of the lines of S.

Let S be an incidence structure. A plane of S is an element of the planes of S.

For simplicity, we adopt the following convention: S is an incidence structure, A, B, C, D are points of S, L is a line of S, P is a plane of S, and F, G are subsets of the points of S.

Let S be a projective incidence structure, let A be a point of S, and let L be a line of S. We say that A lies on L if and only if:

(Def. 1) $\langle A, L\rangle \in$ the incidence of S.

Let us consider S, let A be a point of S, and let P be a plane of S. We say that A lies on P if and only if:

(Def. 2) $\langle A, P\rangle \in$ the incidence2 of S.

Let us consider S, let L be a line of S, and let P be a plane of S. We say that L lies on P if and only if:

(Def. 3) $\langle L, P\rangle \in$ the incidence3 of S.

Let S be a projective incidence structure, let F be a subset of the points of S, and let L be a line of S. We say that F lies on L if and only if:

(Def. 4) For every point A of S such that $A \in F$ holds A lies on L.

Let us consider S, let F be a subset of the points of S, and let P be a plane of S. We say that F lies on P if and only if:

(Def. 5) For every A such that $A \in F$ holds A lies on P.

Let S be a projective incidence structure and let F be a subset of the points of S. We say that F is linear if and only if:

(Def. 6) There exists a line L of S such that F lies on L.

We introduce F is linear as a synonym of F is linear.

Let S be an incidence structure and let F be a subset of the points of S. We say that F is planar if and only if:

(Def. 7) There exists a plane P of S such that F lies on P.

We introduce F is planar as a synonym of F is planar.

We now state a number of propositions:

(11)[1] $\{A,B\}$ lies on L iff A lies on L and B lies on L.

(12) $\{A,B,C\}$ lies on L iff A lies on L and B lies on L and C lies on L.

(13) $\{A,B\}$ lies on P iff A lies on P and B lies on P.

(14) $\{A,B,C\}$ lies on P iff A lies on P and B lies on P and C lies on P.

(15) $\{A,B,C,D\}$ lies on P iff A lies on P and B lies on P and C lies on P and D lies on P.

(16) If $G \subseteq F$ and F lies on L, then G lies on L.

(17) If $G \subseteq F$ and F lies on P, then G lies on P.

(18) F lies on L and A lies on L iff $F \cup \{A\}$ lies on L.

(19) F lies on P and A lies on P iff $F \cup \{A\}$ lies on P.

(20) $F \cup G$ lies on L iff F lies on L and G lies on L.

(21) $F \cup G$ lies on P iff F lies on P and G lies on P.

(22) If $G \subseteq F$ and F is linear, then G is linear.

(23) If $G \subseteq F$ and F is planar, then G is planar.

Let I_1 be an incidence structure. We say that I_1 is incidence space-like if and only if the conditions (Def. 8) are satisfied.

(Def. 8) For every line L of I_1 there exist points A, B of I_1 such that $A \neq B$ and $\{A,B\}$ lies on L and for all points A, B of I_1 there exists a line L of I_1 such that $\{A,B\}$ lies on L and for all points A, B of I_1 and for all lines K, L of I_1 such that $A \neq B$ and $\{A,B\}$ lies on K and $\{A,B\}$ lies on L holds $K = L$ and for every plane P of I_1 there exists a point A of I_1 such that A lies on P and for all points A, B, C of I_1 there exists a plane P of I_1 such that $\{A,B,C\}$ lies on P and for all points A, B, C of I_1 and for all planes P, Q of I_1 such that $\{A,B,C\}$ is not linear and $\{A,B,C\}$ lies on P and $\{A,B,C\}$ lies on Q holds $P = Q$ and for every line L of I_1 and for every plane P of I_1 such that there exist points A, B of I_1 such that $A \neq B$ and $\{A,B\}$ lies on L and $\{A,B\}$ lies on P holds L lies on P and for every point A of I_1 and for all planes P, Q of I_1 such that A lies on P and A lies on Q there exists a point B of I_1 such that $A \neq B$ and B lies on P and B lies on Q and there exist points A, B, C, D of I_1 such that $\{A,B,C,D\}$ is not planar and for every point A of I_1 and for every line L of I_1 and for every plane P of I_1 such that A lies on L and L lies on P holds A lies on P.

[1] The propositions (1)–(10) have been removed.

Let us observe that there exists an incidence structure which is strict and incidence space-like.

An incidence space is an incidence space-like incidence structure.

For simplicity, we adopt the following convention: S denotes an incidence space, A, B, C, D denote points of S, K, L, L_1, L_2 denote lines of S, P, Q denote planes of S, and F denotes a subset of the points of S.

One can prove the following propositions:

(35)[2] If F lies on L and L lies on P, then F lies on P.

(36) $\{A,A,B\}$ is linear.

(37) $\{A,A,B,C\}$ is planar.

(38) If $\{A,B,C\}$ is linear, then $\{A,B,C,D\}$ is planar.

(39) If $A \neq B$ and $\{A,B\}$ lies on L and C does not lie on L, then $\{A,B,C\}$ is not linear.

(40) If $\{A,B,C\}$ is not linear and $\{A,B,C\}$ lies on P and D does not lie on P, then $\{A,B,C,D\}$ is not planar.

(41) If it is not true that there exists P such that K lies on P and L lies on P, then $K \neq L$.

(42) Suppose that

(i) it is not true that there exists P such that L lies on P and L_1 lies on P and L_2 lies on P, and

(ii) there exists A such that A lies on L and A lies on L_1 and A lies on L_2.

Then $L \neq L_1$.

(43) Suppose L_1 lies on P and L_2 lies on P and L does not lie on P and $L_1 \neq L_2$. Then it is not true that there exists Q such that L lies on Q and L_1 lies on Q and L_2 lies on Q.

(44) There exists P such that A lies on P and L lies on P.

(45) If there exists A such that A lies on K and A lies on L, then there exists P such that K lies on P and L lies on P.

(46) If $A \neq B$, then there exists L such that for every K holds $\{A,B\}$ lies on K iff $K = L$.

(47) If $\{A,B,C\}$ is not linear, then there exists P such that for every Q holds $\{A,B,C\}$ lies on Q iff $P = Q$.

(48) If A does not lie on L, then there exists P such that for every Q holds A lies on Q and L lies on Q iff $P = Q$.

(49) Suppose $K \neq L$ and there exists A such that A lies on K and A lies on L. Then there exists P such that for every Q holds K lies on Q and L lies on Q iff $P = Q$.

Let us consider S and let us consider A, B. Let us assume that $A \neq B$. The functor $\mathrm{Line}(A,B)$ yields a line of S and is defined by:

(Def. 9) $\{A,B\}$ lies on $\mathrm{Line}(A,B)$.

Let us consider S and let us consider A, B, C. Let us assume that $\{A,B,C\}$ is not linear. The functor $\mathrm{Plane}(A,B,C)$ yielding a plane of S is defined as follows:

(Def. 10) $\{A,B,C\}$ lies on $\mathrm{Plane}(A,B,C)$.

Let us consider S and let us consider A, L. Let us assume that A does not lie on L. The functor $\mathrm{Plane}(A,L)$ yielding a plane of S is defined as follows:

(Def. 11) A lies on $\mathrm{Plane}(A,L)$ and L lies on $\mathrm{Plane}(A,L)$.

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[2] The propositions (24)–(34) have been removed.

Let us consider S and let us consider K, L. Let us assume that $K \neq L$ and there exists A such that A lies on K and A lies on L. The functor $\text{Plane}(K,L)$ yielding a plane of S is defined by:

(Def. 12) K lies on $\text{Plane}(K,L)$ and L lies on $\text{Plane}(K,L)$.

The following propositions are true:

(57)[3] If $A \neq B$, then $\text{Line}(A,B) = \text{Line}(B,A)$.

(58) If $\{A,B,C\}$ is not linear, then $\text{Plane}(A,B,C) = \text{Plane}(A,C,B)$.

(59) If $\{A,B,C\}$ is not linear, then $\text{Plane}(A,B,C) = \text{Plane}(B,A,C)$.

(60) If $\{A,B,C\}$ is not linear, then $\text{Plane}(A,B,C) = \text{Plane}(B,C,A)$.

(61) If $\{A,B,C\}$ is not linear, then $\text{Plane}(A,B,C) = \text{Plane}(C,A,B)$.

(62) If $\{A,B,C\}$ is not linear, then $\text{Plane}(A,B,C) = \text{Plane}(C,B,A)$.

(64)[4] If $K \neq L$ and there exists A such that A lies on K and A lies on L, then $\text{Plane}(K,L) = \text{Plane}(L,K)$.

(65) If $A \neq B$ and C lies on $\text{Line}(A,B)$, then $\{A,B,C\}$ is linear.

(66) If $A \neq B$ and $A \neq C$ and $\{A,B,C\}$ is linear, then $\text{Line}(A,B) = \text{Line}(A,C)$.

(67) If $\{A,B,C\}$ is not linear, then $\text{Plane}(A,B,C) = \text{Plane}(C,\text{Line}(A,B))$.

(68) If $\{A,B,C\}$ is not linear and D lies on $\text{Plane}(A,B,C)$, then $\{A,B,C,D\}$ is planar.

(69) If C does not lie on L and $\{A,B\}$ lies on L and $A \neq B$, then $\text{Plane}(C,L) = \text{Plane}(A,B,C)$.

(70) If $\{A,B,C\}$ is not linear, then $\text{Plane}(A,B,C) = \text{Plane}(\text{Line}(A,B),\text{Line}(A,C))$.

(71) There exist A, B, C such that $\{A,B,C\}$ lies on P and $\{A,B,C\}$ is not linear.

(72) There exist A, B, C, D such that A lies on P and $\{A,B,C,D\}$ is not planar.

(73) There exists B such that $A \neq B$ and B lies on L.

(74) If $A \neq B$, then there exists C such that C lies on P and $\{A,B,C\}$ is not linear.

(75) If $\{A,B,C\}$ is not linear, then there exists D such that $\{A,B,C,D\}$ is not planar.

(76) There exist B, C such that $\{B,C\}$ lies on P and $\{A,B,C\}$ is not linear.

(77) If $A \neq B$, then there exist C, D such that $\{A,B,C,D\}$ is not planar.

(78) There exist B, C, D such that $\{A,B,C,D\}$ is not planar.

(79) There exists L such that A does not lie on L and L lies on P.

(80) Suppose A lies on P. Then there exist L, L_1, L_2 such that $L_1 \neq L_2$ and L_1 lies on P and L_2 lies on P and L does not lie on P and A lies on L and A lies on L_1 and A lies on L_2.

(81) There exist L, L_1, L_2 such that

(i) A lies on L,

(ii) A lies on L_1,

(iii) A lies on L_2, and

(iv) it is not true that there exists P such that L lies on P and L_1 lies on P and L_2 lies on P.

(82) There exists P such that A lies on P and L does not lie on P.

[3] The propositions (50)–(56) have been removed.

[4] The proposition (63) has been removed.

(83) There exists A such that A lies on P and A does not lie on L.

(84) It is not true that there exists K and there exists P such that L lies on P and K lies on P.

(85) There exist P, Q such that $P \neq Q$ and L lies on P and L lies on Q.

(87)[5] If L does not lie on P and $\{A,B\}$ lies on L and $\{A,B\}$ lies on P, then $A = B$.

(88) Suppose $P \neq Q$. Then
 (i) it is not true that there exists A such that A lies on P and A lies on Q, or
 (ii) there exists L such that for every B holds B lies on P and B lies on Q iff B lies on L.

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[5] The proposition (86) has been removed.