JOURNAL OF FORMALIZED MATHEMATICS
Volume 10, Released 1998, Published 2003
Inst. of Computer Science, Univ. of Białystok

Graph Theoretical Properties of Arcs in the Plane and Fashoda Meet Theorem

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Summary. We define a graph on an abstract set, edges of which are pairs of any two elements. For any finite sequence of a plane, we give a definition of nodic, which means that edges by a finite sequence are crossed only at terminals. If the first point and the last point of a finite sequence differs, simpleness as a chain and nodic condition imply unfoldedness and s.n.c. condition. We generalize Goboard Theorem, proved by us before, to a continuous case. We call this Fashoda Meet Theorem, which was taken from Fashoda incident of 100 years ago.

MML Identifier: `JGRAPH_1`.

WWW: `http://mizar.org/JFM/Vol10/jgraph_1.html`

The articles [32], [10], [36], [3], [33], [20], [37], [8], [9], [4], [11], [16], [1], [2], [21], [28], [25], [35], [26], [19], [27], [29], [24], [23], [18], [6], [14], [5], [15], [22], [30], [34], [13], [12], [31], [17], and [7] provide the notation and terminology for this paper.

1. A Graph by Cartesian Product

For simplicity, we follow the rules: G is a graph, v_1 is a finite sequence of elements of the vertices of G, I_1 is an oriented chain of G, n, m, k, i, j are natural numbers, and r, r_1, r_2 are real numbers.

One can prove the following propositions:

(2)[1] $\sqrt{{r_1}^2+{r_2}^2} \leq |r_1|+|r_2|$.

(3) $|r_1| \leq \sqrt{{r_1}^2+{r_2}^2}$ and $|r_2| \leq \sqrt{{r_1}^2+{r_2}^2}$.

(4) Let given v_1. Suppose I_1 is Simple and v_1 is oriented vertex seq of I_1. Let given n, m. If $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} v_1$ and $v_1(n) = v_1(m)$, then $n = 1$ and $m = \operatorname{len} v_1$.

Let X be a set. The functor PGraph X yielding a multi graph structure is defined by:

(Def. 1) $\mathrm{PGraph}\,X = \langle X, [: X, X :], \pi_1(X \times X), \pi_2(X \times X)\rangle$.

One can prove the following propositions:

(5) For every non empty set X holds PGraph X is a graph.

(6) For every set X holds the vertices of $\mathrm{PGraph}\,X = X$.

[1] The proposition (1) has been removed.

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Let f be a finite sequence. The functor $\mathrm{PairF}\,f$ yielding a finite sequence is defined as follows:

(Def. 2) $\mathrm{len}\,\mathrm{PairF}\,f = \mathrm{len}\,f -' 1$ and for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\,f$ holds $(\mathrm{PairF}\,f)(i) = \langle f(i),\ f(i+1)\rangle$.

In the sequel X is a non empty set.

Let X be a non empty set. Observe that $\mathrm{PGraph}\,X$ is graph-like.

Next we state two propositions:

(7) Every finite sequence of elements of X is a finite sequence of elements of the vertices of $\mathrm{PGraph}\,X$.

(8) For every finite sequence f of elements of X holds $\mathrm{PairF}\,f$ is a finite sequence of elements of the edges of $\mathrm{PGraph}\,X$.

Let X be a non empty set and let f be a finite sequence of elements of X. Then $\mathrm{PairF}\,f$ is a finite sequence of elements of the edges of $\mathrm{PGraph}\,X$.

Next we state two propositions:

(9) Let n be a natural number and f be a finite sequence of elements of X. If $1 \leq n$ and $n \leq \mathrm{len}\,\mathrm{PairF}\,f$, then $(\mathrm{PairF}\,f)(n) \in$ the edges of $\mathrm{PGraph}\,X$.

(10) For every finite sequence f of elements of X holds $\mathrm{PairF}\,f$ is an oriented chain of $\mathrm{PGraph}\,X$.

Let X be a non empty set and let f be a finite sequence of elements of X. Then $\mathrm{PairF}\,f$ is an oriented chain of $\mathrm{PGraph}\,X$.

One can prove the following proposition

(11) Let f be a finite sequence of elements of X and f_1 be a finite sequence of elements of the vertices of $\mathrm{PGraph}\,X$. If $\mathrm{len}\,f \geq 1$ and $f = f_1$, then f_1 is oriented vertex seq of $\mathrm{PairF}\,f$.

2. Shortcuts of Finite Sequences in Plane

Let X be a non empty set and let f, g be finite sequences of elements of X. We say that g is Shortcut of f if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $f(1) = g(1)$,

(ii) $f(\mathrm{len}\,f) = g(\mathrm{len}\,g)$, and

(iii) there exists a FinSubsequence f_2 of $\mathrm{PairF}\,f$ and there exists a FinSubsequence f_3 of f and there exists an oriented simple chain s_1 of $\mathrm{PGraph}\,X$ and there exists a finite sequence g_1 of elements of the vertices of $\mathrm{PGraph}\,X$ such that $\mathrm{Seq}\,f_2 = s_1$ and $\mathrm{Seq}\,f_3 = g$ and $g_1 = g$ and g_1 is oriented vertex seq of s_1.

Next we state four propositions:

(12) For all finite sequences f, g of elements of X such that g is Shortcut of f holds $1 \leq \mathrm{len}\,g$ and $\mathrm{len}\,g \leq \mathrm{len}\,f$.

(13) Let f be a finite sequence of elements of X. Suppose $\mathrm{len}\,f \geq 1$. Then there exists a finite sequence g of elements of X such that g is Shortcut of f.

(14) For all finite sequences f, g of elements of X such that g is Shortcut of f holds $\mathrm{rng}\,\mathrm{PairF}\,g \subseteq \mathrm{rng}\,\mathrm{PairF}\,f$.

(15) Let f, g be finite sequences of elements of X. Suppose $f(1) \neq f(\mathrm{len}\,f)$ and g is Shortcut of f. Then g is one-to-one and $\mathrm{rng}\,\mathrm{PairF}\,g \subseteq \mathrm{rng}\,\mathrm{PairF}\,f$ and $g(1) = f(1)$ and $g(\mathrm{len}\,g) = f(\mathrm{len}\,f)$.

Let us consider n and let I_1 be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$. We say that I_1 is nodic if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let given i, j. Suppose $\mathcal{L}(I_1,i)$ meets $\mathcal{L}(I_1,j)$. Then $\mathcal{L}(I_1,i)\cap\mathcal{L}(I_1,j)=\{I_1(i)\}$ but $I_1(i)=I_1(j)$ or $I_1(i)=I_1(j+1)$ or $\mathcal{L}(I_1,i)\cap\mathcal{L}(I_1,j)=\{I_1(i+1)\}$ but $I_1(i+1)=I_1(j)$ or $I_1(i+1)=I_1(j+1)$ or $\mathcal{L}(I_1,i)=\mathcal{L}(I_1,j)$.

Next we state a number of propositions:

(16) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ such that f is s.n.c. holds f is s.c.c..

(17) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ such that f is s.c.c. and $\mathcal{L}(f,1)$ misses $\mathcal{L}(f,\operatorname{len} f -' 1)$ holds f is s.n.c..

(18) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ such that f is nodic and PairF f is Simple holds f is s.c.c..

(19) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ such that f is nodic and PairF f is Simple and $f(1)\neq f(\operatorname{len} f)$ holds f is s.n.c..

(20) For all points p_1, p_2, p_3 of $\mathcal{E}^n_{\rm T}$ such that there exists a set x such that $x\neq p_2$ and $x\in\mathcal{L}(p_1,p_2)\cap\mathcal{L}(p_2,p_3)$ holds $p_1\in\mathcal{L}(p_2,p_3)$ or $p_3\in\mathcal{L}(p_1,p_2)$.

(21) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is s.n.c. and $\mathcal{L}(f,1)\cap\mathcal{L}(f,1+1)\subseteq\{f_{1+1}\}$ and $\mathcal{L}(f,\operatorname{len} f -' 2)\cap\mathcal{L}(f,\operatorname{len} f -' 1)\subseteq\{f_{\operatorname{len} f -' 1}\}$. Then f is unfolded.

(22) For every finite sequence f of elements of X such that PairF f is Simple and $f(1)\neq f(\operatorname{len} f)$ holds f is one-to-one and $\operatorname{len} f\neq 1$.

(23) For every finite sequence f of elements of X such that f is one-to-one and $\operatorname{len} f>1$ holds PairF f is Simple and $f(1)\neq f(\operatorname{len} f)$.

(24) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. If f is nodic and PairF f is Simple and $f(1)\neq f(\operatorname{len} f)$, then f is unfolded.

(25) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$ and given i. Suppose g is Shortcut of f and $1\leq i$ and $i+1\leq \operatorname{len} g$. Then there exists a natural number k_1 such that $1\leq k_1$ and $k_1+1\leq\operatorname{len} f$ and $f_{k_1}=g_i$ and $f_{k_1+1}=g_{i+1}$ and $f(k_1)=g(i)$ and $f(k_1+1)=g(i+1)$.

(26) For all finite sequences f, g of elements of $\mathcal{E}^2_{\rm T}$ such that g is Shortcut of f holds $\operatorname{rng} g\subseteq\operatorname{rng} f$.

(27) For all finite sequences f, g of elements of $\mathcal{E}^2_{\rm T}$ such that g is Shortcut of f holds $\widetilde{\mathcal{L}}(g)\subseteq\widetilde{\mathcal{L}}(f)$.

(28) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. If f is special and g is Shortcut of f, then g is special.

(29) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Suppose f is special and $2\leq\operatorname{len} f$ and $f(1)\neq f(\operatorname{len} f)$. Then there exists a finite sequence g of elements of $\mathcal{E}^2_{\rm T}$ such that $2\leq\operatorname{len} g$ and g is special and one-to-one and $\widetilde{\mathcal{L}}(g)\subseteq\widetilde{\mathcal{L}}(f)$ and $f(1)=g(1)$ and $f(\operatorname{len} f)=g(\operatorname{len} g)$ and $\operatorname{rng} g\subseteq\operatorname{rng} f$.

(30) Let f_1, f_4 be finite sequences of elements of $\mathcal{E}^2_{\rm T}$. Suppose that f_1 is special and f_4 is special and $2\leq\operatorname{len} f_1$ and $2\leq\operatorname{len} f_4$ and $f_1(1)\neq f_1(\operatorname{len} f_1)$ and $f_4(1)\neq f_4(\operatorname{len} f_4)$ and $\mathbf{X}$-coordinate(f_1) lies between $(\mathbf{X}$-coordinate$(f_1))(1)$ and $(\mathbf{X}$-coordinate$(f_1))(\operatorname{len} f_1)$ and $\mathbf{X}$-coordinate(f_4) lies between $(\mathbf{X}$-coordinate$(f_1))(1)$ and $(\mathbf{X}$-coordinate$(f_1))(\operatorname{len} f_1)$ and $\mathbf{Y}$-coordinate(f_1) lies between $(\mathbf{Y}$-coordinate$(f_4))(1)$ and $(\mathbf{Y}$-coordinate$(f_4))(\operatorname{len} f_4)$ and $\mathbf{Y}$-coordinate(f_4) lies between $(\mathbf{Y}$-coordinate$(f_4))(1)$ and $(\mathbf{Y}$-coordinate$(f_4))(\operatorname{len} f_4)$. Then $\widetilde{\mathcal{L}}(f_1)$ meets $\widetilde{\mathcal{L}}(f_4)$.

3. Norm of Points in $\mathcal{E}^n_{\rm T}$

The following proposition is true

(31) For all real numbers a, b, r_1, r_2 such that $a \leq r_1$ and $r_1 \leq b$ and $a \leq r_2$ and $r_2 \leq b$ holds $|r_1 - r_2| \leq b - a$.

Let us consider n and let p be a point of $\mathcal{E}^n_{\rm T}$. Then $|p|$ can be characterized by the condition:

(Def. 5) For every element w of $\mathcal{R}^n$ such that $p = w$ holds $|p| = |w|$.

In the sequel p, p_1, p_2 denote points of $\mathcal{E}^n_{\rm T}$.
We now state several propositions:

(45)[2] For all points x_1, x_2 of $\mathcal{E}^n$ such that $x_1 = p_1$ and $x_2 = p_2$ holds $|p_1 - p_2| = \rho(x_1, x_2)$.

(46) For every point p of $\mathcal{E}^2_{\rm T}$ holds $|p|^{\mathbf{2}} = (p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}$.

(47) For every point p of $\mathcal{E}^2_{\rm T}$ holds $|p| = \sqrt{(p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}}$.

(48) For every point p of $\mathcal{E}^2_{\rm T}$ holds $|p| \leq |p_{\mathbf{1}}| + |p_{\mathbf{2}}|$.

(49) For all points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ holds $|p_1 - p_2| \leq |(p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}| + |(p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}|$.

(50) For every point p of $\mathcal{E}^2_{\rm T}$ holds $|p_{\mathbf{1}}| \leq |p|$ and $|p_{\mathbf{2}}| \leq |p|$.

(51) For all points p_1, p_2 of $\mathcal{E}^2_{\rm T}$ holds $|(p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}| \leq |p_1 - p_2|$ and $|(p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}| \leq |p_1 - p_2|$.

(52) If $p \in \mathcal{L}(p_1, p_2)$, then there exists r such that $0 \leq r$ and $r \leq 1$ and $p = (1 - r) \cdot p_1 + r \cdot p_2$.

(53) If $p \in \mathcal{L}(p_1, p_2)$, then $|p - p_1| \leq |p_1 - p_2|$ and $|p - p_2| \leq |p_1 - p_2|$.

4. Extended Goboard Theorem and Fashoda Meet Theorem

In the sequel M denotes a non empty metric space.
Next we state several propositions:

(54) For all subsets P, Q of $M_{\rm top}$ such that $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact holds $\text{dist}^{\min}_{\min}(P, Q) \geq 0$.

(55) Let P, Q be subsets of $M_{\rm top}$. Suppose $P \neq \emptyset$ and P is compact and $Q \neq \emptyset$ and Q is compact. Then P misses Q if and only if $\text{dist}^{\min}_{\min}(P, Q) > 0$.

(56) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and a, c, d be real numbers. Suppose that

(i) $1 \leq \text{len}\, f$,

(ii) **X**-coordinate(f) lies between (**X**-coordinate$(f))(1)$ and (**X**-coordinate$(f))(\text{len}\, f)$,

(iii) **Y**-coordinate(f) lies between c and d,

(iv) $a > 0$, and

(v) for every i such that $1 \leq i$ and $i + 1 \leq \text{len}\, f$ holds $|f_i - f_{i+1}| < a$.

Then there exists a finite sequence g of elements of $\mathcal{E}^2_{\rm T}$ such that

g is special and $g(1) = f(1)$ and $g(\text{len}\, g) = f(\text{len}\, f)$ and $\text{len}\, g \geq \text{len}\, f$ and **X**-coordinate(g) lies between (**X**-coordinate$(f))(1)$ and (**X**-coordinate$(f))(\text{len}\, f)$ and **Y**-coordinate(g) lies between c and d and for every j such that $j \in \text{dom}\, g$ there exists k such that $k \in \text{dom}\, f$ and $|g_j - f_k| < a$ and for every j such that $1 \leq j$ and $j + 1 \leq \text{len}\, g$ holds $|g_j - g_{j+1}| < a$.

[2] The propositions (32)–(44) have been removed.

(57) Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$ and a, c, d be real numbers. Suppose that

(i) $1 \leq \operatorname{len} f$,

(ii) $\mathbf{Y}$-coordinate(f) lies between $(\mathbf{Y}\text{-coordinate}(f))(1)$ and $(\mathbf{Y}\text{-coordinate}(f))(\operatorname{len} f)$,

(iii) $\mathbf{X}$-coordinate(f) lies between c and d,

(iv) $a > 0$, and

(v) for every i such that $1 \leq i$ and $i+1 \leq \operatorname{len} f$ holds $|f_i - f_{i+1}| < a$.

Then there exists a finite sequence g of elements of $\mathcal{E}^2_{\rm T}$ such that

g is special and $g(1) = f(1)$ and $g(\operatorname{len} g) = f(\operatorname{len} f)$ and $\operatorname{len} g \geq \operatorname{len} f$ and $\mathbf{Y}$-coordinate(g) lies between $(\mathbf{Y}\text{-coordinate}(f))(1)$ and $(\mathbf{Y}\text{-coordinate}(f))(\operatorname{len} f)$ and $\mathbf{X}$-coordinate(g) lies between c and d and for every j such that $j \in \operatorname{dom} g$ there exists k such that $k \in \operatorname{dom} f$ and $|g_j - f_k| < a$ and for every j such that $1 \leq j$ and $j+1 \leq \operatorname{len} g$ holds $|g_j - g_{j+1}| < a$.

(59)[3] For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ such that $1 \leq \operatorname{len} f$ holds $\operatorname{len} \mathbf{X}\text{-coordinate}(f) = \operatorname{len} f$ and $(\mathbf{X}\text{-coordinate}(f))(1) = (f_1)_{\mathbf{1}}$ and $(\mathbf{X}\text{-coordinate}(f))(\operatorname{len} f) = (f_{\operatorname{len} f})_{\mathbf{1}}$.

(60) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ such that $1 \leq \operatorname{len} f$ holds $\operatorname{len} \mathbf{Y}\text{-coordinate}(f) = \operatorname{len} f$ and $(\mathbf{Y}\text{-coordinate}(f))(1) = (f_1)_{\mathbf{2}}$ and $(\mathbf{Y}\text{-coordinate}(f))(\operatorname{len} f) = (f_{\operatorname{len} f})_{\mathbf{2}}$.

(61) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ such that $i \in \operatorname{dom} f$ holds $(\mathbf{X}\text{-coordinate}(f))(i) = (f_i)_{\mathbf{1}}$ and $(\mathbf{Y}\text{-coordinate}(f))(i) = (f_i)_{\mathbf{2}}$.

(62) Let P, Q be non empty subsets of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose that

(i) P is an arc from p_1 to p_2,

(ii) Q is an arc from q_1 to q_2,

(iii) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in P$ holds $(p_1)_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$,

(iv) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in Q$ holds $(p_1)_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (p_2)_{\mathbf{1}}$,

(v) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in P$ holds $(q_1)_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (q_2)_{\mathbf{2}}$, and

(vi) for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in Q$ holds $(q_1)_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (q_2)_{\mathbf{2}}$.

Then P meets Q.

In the sequel X, Y are non empty topological spaces.
The following propositions are true:

(63) Let f be a map from X into Y, P be a non empty subset of Y, and f_1 be a map from X into $Y \restriction P$. If $f = f_1$ and f is continuous, then f_1 is continuous.

(64) Let f be a map from X into Y and P be a non empty subset of Y. Suppose X is compact and Y is a T_2 space and f is continuous and one-to-one and $P = \operatorname{rng} f$. Then there exists a map f_1 from X into $Y \restriction P$ such that $f = f_1$ and f_1 is a homeomorphism.

(65) Let f, g be maps from $\mathbb{I}$ into $\mathcal{E}^2_{\rm T}$, a, b, c, d be real numbers, and O, I be points of $\mathbb{I}$. Suppose that $O = 0$ and $I = 1$ and f is continuous and one-to-one and g is continuous and one-to-one and $f(O)_{\mathbf{1}} = a$ and $f(I)_{\mathbf{1}} = b$ and $g(O)_{\mathbf{2}} = c$ and $g(I)_{\mathbf{2}} = d$ and for every point r of $\mathbb{I}$ holds $a \leq f(r)_{\mathbf{1}}$ and $f(r)_{\mathbf{1}} \leq b$ and $a \leq g(r)_{\mathbf{1}}$ and $g(r)_{\mathbf{1}} \leq b$ and $c \leq f(r)_{\mathbf{2}}$ and $f(r)_{\mathbf{2}} \leq d$ and $c \leq g(r)_{\mathbf{2}}$ and $g(r)_{\mathbf{2}} \leq d$. Then $\operatorname{rng} f$ meets $\operatorname{rng} g$.

[3] The proposition (58) has been removed.

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Received August 21, 1998

Published January 2, 2004