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On the Lattice of Subgroups of a Group

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The articles [10], [5], [17], [2], [18], [9], [4], [3], [19], [11], [8], [13], [15], [12], [14], [16], [1], [7], and [6] provide the notation and terminology for this paper.

The following propositions are true:

(1) Let G be a group and H_1, H_2 be subgroups of G. Then the carrier of $H_1 \cap H_2 =$ (the carrier of H_1) $\cap$ (the carrier of H_2).

(2) For every group G and for every set h holds $h \in \mathrm{SubGr}\, G$ iff there exists a strict subgroup H of G such that $h = H$.

(3) Let G be a group, A be a subset of G, and H be a strict subgroup of G. If $A =$ the carrier of H, then $\mathrm{gr}(A) = H$.

(4) Let G be a group, H_1, H_2 be subgroups of G, and A be a subset of G. If $A =$ (the carrier of H_1) $\cup$ (the carrier of H_2), then $H_1 \sqcup H_2 = \mathrm{gr}(A)$.

(5) For every group G and for all subgroups H_1, H_2 of G and for every element g of G such that $g \in H_1$ or $g \in H_2$ holds $g \in H_1 \sqcup H_2$.

(6) Let G_1, G_2 be groups, f be a homomorphism from G_1 to G_2, and H_1 be a subgroup of G_1. Then there exists a strict subgroup H_2 of G_2 such that the carrier of $H_2 = f^\circ$(the carrier of H_1).

(7) Let G_1, G_2 be groups, f be a homomorphism from G_1 to G_2, and H_2 be a subgroup of G_2. Then there exists a strict subgroup H_1 of G_1 such that the carrier of $H_1 = f^{-1}$(the carrier of H_2).

(10)[1] Let G_1, G_2 be groups, f be a homomorphism from G_1 to G_2, H_1, H_2 be subgroups of G_1, and H_3, H_4 be subgroups of G_2. Suppose the carrier of $H_3 = f^\circ$(the carrier of H_1) and the carrier of $H_4 = f^\circ$(the carrier of H_2). If H_1 is a subgroup of H_2, then H_3 is a subgroup of H_4.

(11) Let G_1, G_2 be groups, f be a homomorphism from G_1 to G_2, H_1, H_2 be subgroups of G_2, and H_3, H_4 be subgroups of G_1. Suppose the carrier of $H_3 = f^{-1}$(the carrier of H_1) and the carrier of $H_4 = f^{-1}$(the carrier of H_2). If H_1 is a subgroup of H_2, then H_3 is a subgroup of H_4.

(12) Let G_1, G_2 be groups, f be a function from the carrier of G_1 into the carrier of G_2, and A be a subset of G_1. Then $f^\circ A \subseteq f^\circ$(the carrier of $\mathrm{gr}(A)$).

[1] The propositions (8) and (9) have been removed.

(13) Let G_1, G_2 be groups, H_1, H_2 be subgroups of G_1, f be a function from the carrier of G_1 into the carrier of G_2, and A be a subset of G_1. Suppose $A =$ (the carrier of $H_1) \cup$ (the carrier of H_2). Then f°(the carrier of $H_1 \sqcup H_2) = f^\circ$(the carrier of gr$(A)$).

(14) For every group G and for every subset A of G such that $A = \{1_G\}$ holds $\mathrm{gr}(A) = \{\mathbf{1}\}_G$.

Let G be a group. The functor $\overline{G}$ yielding a function from SubGr G into $2^{\text{the carrier of } G}$ is defined as follows:

(Def. 1) For every strict subgroup H of G holds $\overline{G}(H) =$ the carrier of H.

One can prove the following propositions:

(18)[2] Let G be a group, H be a strict subgroup of G, and x be an element of G. Then $x \in \overline{G}(H)$ if and only if $x \in H$.

(19) For every group G and for every strict subgroup H of G holds $1_G \in \overline{G}(H)$.

(20) For every group G and for every strict subgroup H of G holds $\overline{G}(H) \neq \emptyset$.

(21) Let G be a group, H be a strict subgroup of G, and g_1, g_2 be elements of G. If $g_1 \in \overline{G}(H)$ and $g_2 \in \overline{G}(H)$, then $g_1 \cdot g_2 \in \overline{G}(H)$.

(22) For every group G and for every strict subgroup H of G and for every element g of G such that $g \in \overline{G}(H)$ holds $g^{-1} \in \overline{G}(H)$.

(23) For every group G and for all strict subgroups H_1, H_2 of G holds the carrier of $H_1 \cap H_2 = \overline{G}(H_1) \cap \overline{G}(H_2)$.

(24) For every group G and for all strict subgroups H_1, H_2 of G holds $\overline{G}(H_1 \cap H_2) = \overline{G}(H_1) \cap \overline{G}(H_2)$.

Let G be a group and let F be a non empty subset of SubGr G. The functor $\bigcap F$ yields a strict subgroup of G and is defined as follows:

(Def. 2) The carrier of $\bigcap F = \bigcap(\overline{G}^\circ F)$.

We now state several propositions:

(25) For every group G and for every non empty subset F of SubGr G such that $\{\mathbf{1}\}_G \in F$ holds $\bigcap F = \{\mathbf{1}\}_G$.

(26) For every group G and for every element h of SubGr G and for every non empty subset F of SubGr G such that $F = \{h\}$ holds $\bigcap F = h$.

(27) Let G be a group, H_1, H_2 be subgroups of G, and h_1, h_2 be elements of $\mathbb{L}_G$. If $h_1 = H_1$ and $h_2 = H_2$, then $h_1 \sqcup h_2 = H_1 \sqcup H_2$.

(28) Let G be a group, H_1, H_2 be subgroups of G, and h_1, h_2 be elements of $\mathbb{L}_G$. If $h_1 = H_1$ and $h_2 = H_2$, then $h_1 \sqcap h_2 = H_1 \cap H_2$.

(29) Let G be a group, p be an element of $\mathbb{L}_G$, and H be a subgroup of G. If $p = H$, then H is a strict subgroup of G.

(30) Let G be a group, H_1, H_2 be subgroups of G, and p, q be elements of $\mathbb{L}_G$. Suppose $p = H_1$ and $q = H_2$. Then $p \sqsubseteq q$ if and only if the carrier of $H_1 \subseteq$ the carrier of H_2.

(31) Let G be a group, H_1, H_2 be subgroups of G, and p, q be elements of $\mathbb{L}_G$. If $p = H_1$ and $q = H_2$, then $p \sqsubseteq q$ iff H_1 is a subgroup of H_2.

(32) For every group G holds $\mathbb{L}_G$ is complete.

[2] The propositions (15)–(17) have been removed.

Let G_1, G_2 be groups and let f be a function from the carrier of G_1 into the carrier of G_2. The functor FuncLatt(f) yielding a function from the carrier of $\mathbb{L}_{(G_1)}$ into the carrier of $\mathbb{L}_{(G_2)}$ is defined as follows:

(Def. 3) For every strict subgroup H of G_1 and for every subset A of G_2 such that $A = f^{\circ}$(the carrier of H) holds $(\mathrm{FuncLatt}(f))(H) = \mathrm{gr}(A)$.

One can prove the following propositions:

(33) For every group G holds $\mathrm{FuncLatt}(\mathrm{id}_{\text{the carrier of } G}) = \mathrm{id}_{\text{the carrier of } \mathbb{L}_G}$.

(34) For all groups G_1, G_2 and for every homomorphism f from G_1 to G_2 such that f is one-to-one holds FuncLatt(f) is one-to-one.

(35) For all groups G_1, G_2 and for every homomorphism f from G_1 to G_2 holds $(\mathrm{FuncLatt}(f))(\{\mathbf{1}\}_{(G_1)}) = \{\mathbf{1}\}_{(G_2)}$.

(36) Let G_1, G_2 be groups and f be a homomorphism from G_1 to G_2. Suppose f is one-to-one. Then FuncLatt(f) is a lower homomorphism between $\mathbb{L}_{(G_1)}$ and $\mathbb{L}_{(G_2)}$.

(37) Let G_1, G_2 be groups and f be a homomorphism from G_1 to G_2. Then FuncLatt(f) is an upper homomorphism between $\mathbb{L}_{(G_1)}$ and $\mathbb{L}_{(G_2)}$.

(38) Let G_1, G_2 be groups and f be a homomorphism from G_1 to G_2. If f is one-to-one, then FuncLatt(f) is a homomorphism from $\mathbb{L}_{(G_1)}$ to $\mathbb{L}_{(G_2)}$.

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