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The Jónsson Theorem

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The articles [23], [22], [14], [27], [28], [30], [29], [11], [12], [5], [25], [2], [18], [3], [4], [1], [24], [21], [13], [16], [26], [19], [17], [9], [15], [31], [6], [10], [7], [8], [32], and [20] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *RecChoice* deals with a set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a function f such that $\mathrm{dom}\, f = \mathbb{N}$ and $f(0) = \mathcal{A}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1)]$

provided the parameters satisfy the following condition:

- For every natural number n and for every set x there exists a set y such that $\mathcal{P}[n,x,y]$.

One can prove the following three propositions:

(1) For every function f and for every function yielding function F such that $f = \bigcup \mathrm{rng} F$ holds $\mathrm{dom}\, f = \bigcup \mathrm{rng}(\mathrm{dom}_\kappa F(\kappa))$.

(2) For all non empty sets A, B holds $[:\bigcup A, \bigcup B:] = \bigcup \{[:a, b:]; a$ ranges over elements of A, b ranges over elements of B: $a \in A \wedge b \in B\}$.

(3) For every non empty set A such that A is $\subseteq$-linear holds $[:\bigcup A, \bigcup A:] = \bigcup \{[:a, a:]; a$ ranges over elements of A: $a \in A\}$.

2. An Equivalence Lattice of a Set

In the sequel X is a non empty set.

Let A be a set. The functor $\mathrm{EqRelPoset}(A)$ yields a poset and is defined as follows:

(Def. 1) $\mathrm{EqRelPoset}(A) = \mathrm{Poset}(\mathrm{EqRelLatt}(A))$.

Let A be a set. Observe that $\mathrm{EqRelPoset}(A)$ has g.l.b.'s and l.u.b.'s.

We now state several propositions:

(4) For all sets A, x holds $x \in$ the carrier of $\mathrm{EqRelPoset}(A)$ iff x is an equivalence relation of A.

(5) For every set A and for all elements x, y of $\mathrm{EqRelLatt}(A)$ holds $x \sqsubseteq y$ iff $x \subseteq y$.

(6) For every set A and for all elements a, b of $\mathrm{EqRelPoset}(A)$ holds $a \leq b$ iff $a \subseteq b$.

(7) For every lattice L and for all elements a, b of $\mathrm{Poset}(L)$ holds $a \sqcap b = {}^\bullet a \sqcap {}^\bullet b$.

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(8) For every set A and for all elements a, b of EqRelPoset(A) holds $a \sqcap b = a \cap b$.

(9) For every lattice L and for all elements a, b of Poset(L) holds $a \sqcup b = {}^{\bullet}a \sqcup {}^{\bullet}b$.

(10) Let A be a set, a, b be elements of EqRelPoset(A), and E_1, E_2 be equivalence relations of A. If $a = E_1$ and $b = E_2$, then $a \sqcup b = E_1 \sqcup E_2$.

Let L be a non empty relational structure. Let us observe that L is complete if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let X be a subset of L. Then there exists an element a of L such that $a \leq X$ and for every element b of L such that $b \leq X$ holds $b \leq a$.

Let A be a set. Observe that EqRelPoset(A) is complete.

3. A Type of a Sublattice of Equivalence Lattice of a Set

Let L_1, L_2 be lattices. Note that there exists a map from L_1 into L_2 which is meet-preserving and join-preserving.

Let L_1, L_2 be lattices. A homomorphism from L_1 to L_2 is a meet-preserving join-preserving map from L_1 into L_2.

Let L be a lattice. Observe that there exists a relational substructure of L which is meet-inheriting, join-inheriting, and strict.

Let L_1, L_2 be lattices and let f be a homomorphism from L_1 to L_2. Then Im f is a strict full sublattice of L_2.

We adopt the following convention: e, e_1, e_2 denote equivalence relations of X and x, y denote sets.

Let us consider X, let f be a non empty finite sequence of elements of X, let us consider x, y, and let R_1, R_2 be binary relations. We say that x and y are joint by f, R_1 and R_2 if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) $f(1) = x$,

(ii) $f(\operatorname{len} f) = y$, and

(iii) for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds if i is odd, then $\langle f(i), f(i+1)\rangle \in R_1$ and if i is even, then $\langle f(i), f(i+1)\rangle \in R_2$.

Next we state two propositions:

(12)[1] Let x be a set, o be a natural number, R_1, R_2 be binary relations, and f be a non empty finite sequence of elements of X. Suppose R_1 is reflexive in X and R_2 is reflexive in X and $f = o \longmapsto x$. Then x and x are joint by f, R_1 and R_2.

(14)[2] Let x, y be sets, R_1, R_2 be binary relations, and n, m be natural numbers. Suppose that

(i) $n \leq m$,

(ii) R_1 is reflexive in X,

(iii) R_2 is reflexive in X, and

(iv) there exists a non empty finite sequence f of elements of X such that $\operatorname{len} f = n$ and x and y are joint by f, R_1 and R_2.

Then there exists a non empty finite sequence h of elements of X such that $\operatorname{len} h = m$ and x and y are joint by h, R_1 and R_2.

Let us consider X and let Y be a sublattice of EqRelPoset(X). Let us assume that there exists e such that $e \in$ the carrier of Y $e \neq \mathrm{id}_X$. And let us assume that there exists a natural number o such that for all e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of X such that $\operatorname{len} F = o$ and x and y are joint by F, e_1 and e_2. The type of Y is a natural number and is defined by the conditions (Def. 4).

[1] The proposition (11) has been removed.
[2] The proposition (13) has been removed.

(Def. 4)(i) For all e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of X such that $\operatorname{len} F =$ (the type of $Y) + 2$ and x and y are joint by F, e_1 and e_2, and

(ii) there exist e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ and it is not true that there exists a non empty finite sequence F of elements of X such that $\operatorname{len} F =$ (the type of $Y) + 1$ and x and y are joint by F, e_1 and e_2.

One can prove the following proposition

(15) Let Y be a sublattice of EqRelPoset(X) and n be a natural number. Suppose that

(i) there exists e such that $e \in$ the carrier of Y and $e \neq \mathrm{id}_X$, and

(ii) for all e_1, e_2, x, y such that $e_1 \in$ the carrier of Y and $e_2 \in$ the carrier of Y and $\langle x, y\rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of X such that $\operatorname{len} F = n + 2$ and x and y are joint by F, e_1 and e_2.

Then the type of $Y \leq n$.

4. A Meet-representation of a Lattice

In the sequel A is a non empty set and L is a lower-bounded lattice.

Let us consider A, L. A bifunction from A into L is a function from $[:A, A:]$ into the carrier of L.

Let us consider A, L, let f be a bifunction from A into L, and let x, y be elements of A. Then $f(x, y)$ is an element of L.

Let us consider A, L and let f be a bifunction from A into L. We say that f is symmetric if and only if:

(Def. 6)[3] For all elements x, y of A holds $f(x, y) = f(y, x)$.

We say that f is zeroed if and only if:

(Def. 7) For every element x of A holds $f(x, x) = \perp_L$.

We say that f satisfies triangle inequality if and only if:

(Def. 8) For all elements x, y, z of A holds $f(x, y) \sqcup f(y, z) \geq f(x, z)$.

Let us consider A, L. One can check that there exists a bifunction from A into L which is symmetric and zeroed and satisfies triangle inequality.

Let us consider A, L. A distance function of A, L is a symmetric zeroed bifunction from A into L satisfying triangle inequality.

Let us consider A, L and let d be a distance function of A, L. The functor $\alpha(d)$ yields a map from L into EqRelPoset(A) and is defined by the condition (Def. 9).

(Def. 9) Let e be an element of L. Then there exists an equivalence relation E of A such that $E = (\alpha(d))(e)$ and for all elements x, y of A holds $\langle x, y\rangle \in E$ iff $d(x, y) \leq e$.

We now state two propositions:

(16) For every distance function d of A, L holds $\alpha(d)$ is meet-preserving.

(17) For every distance function d of A, L such that d is onto holds $\alpha(d)$ is one-to-one.

[3] The definition (Def. 5) has been removed.

5. THE JÓNSSON THEOREM

Let A be a set. The functor A^* is defined by:

(Def. 10) $A^* = A \cup \{\{A\}, \{\{A\}\}, \{\{\{A\}\}\}\}$.

Let A be a set. One can verify that A^* is non empty.

Let us consider A, L, let d be a bifunction from A into L, and let q be an element of $[:A, A,$ the carrier of L, the carrier of $L:]$. The functor d_q^* yields a bifunction from A^* into L and is defined by the conditions (Def. 11).

(Def. 11) For all elements u, v of A holds $d_q^*(u, v) = d(u, v)$ and $d_q^*(\{A\}, \{A\}) = \bot_L$ and $d_q^*(\{\{A\}\}, \{\{A\}\}) = \bot_L$ and $d_q^*(\{\{\{A\}\}\}, \{\{\{A\}\}\}) = \bot_L$ and $d_q^*(\{\{A\}\}, \{\{\{A\}\}\}) = q_{\mathbf{3}}$ and $d_q^*(\{\{\{A\}\}\}, \{\{A\}\}) = q_{\mathbf{3}}$ and $d_q^*(\{A\}, \{\{A\}\}) = q_{\mathbf{4}}$ and $d_q^*(\{\{A\}\}, \{A\}) = q_{\mathbf{4}}$ and $d_q^*(\{A\}, \{\{\{A\}\}\}) = q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$ and $d_q^*(\{\{\{A\}\}\}, \{A\}) = q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$ and for every element u of A holds $d_q^*(u, \{A\}) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}}$ and $d_q^*(\{A\}, u) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}}$ and $d_q^*(u, \{\{A\}\}) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$ and $d_q^*(\{\{A\}\}, u) = d(u, q_{\mathbf{1}}) \sqcup q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$ and $d_q^*(u, \{\{\{A\}\}\}) = d(u, q_{\mathbf{2}}) \sqcup q_{\mathbf{4}}$ and $d_q^*(\{\{\{A\}\}\}, u) = d(u, q_{\mathbf{2}}) \sqcup q_{\mathbf{4}}$.

Next we state several propositions:

(18) Let d be a bifunction from A into L. Suppose d is zeroed. Let q be an element of $[:A, A,$ the carrier of L, the carrier of $L:]$. Then d_q^* is zeroed.

(19) Let d be a bifunction from A into L. Suppose d is symmetric. Let q be an element of $[:A, A,$ the carrier of L, the carrier of $L:]$. Then d_q^* is symmetric.

(20) Let d be a bifunction from A into L. Suppose d is symmetric and satisfies triangle inequality. Let q be an element of $[:A, A,$ the carrier of L, the carrier of $L:]$. If $d(q_{\mathbf{1}}, q_{\mathbf{2}}) \leq q_{\mathbf{3}} \sqcup q_{\mathbf{4}}$, then d_q^* satisfies triangle inequality.

(21) For every set A holds $A \subseteq A^*$.

(22) Let d be a bifunction from A into L and q be an element of $[:A, A,$ the carrier of L, the carrier of $L:]$. Then $d \subseteq d_q^*$.

Let us consider A, L and let d be a bifunction from A into L. The functor $\mathrm{DistEsti}(d)$ yielding a cardinal number is defined as follows:

(Def. 12) $\mathrm{DistEsti}(d) \approx \{\langle x, y, a, b \rangle; x$ ranges over elements of A, y ranges over elements of A, a ranges over elements of L, b ranges over elements of L: $d(x, y) \leq a \sqcup b\}$.

Next we state the proposition

(23) For every distance function d of A, L holds $\mathrm{DistEsti}(d) \neq \emptyset$.

In the sequel T is a transfinite sequence and O, O_1, O_2 are ordinal numbers.

Let us consider A and let us consider O. The functor $\mathrm{ConsecutiveSet}(A, O)$ is defined by the condition (Def. 13).

(Def. 13) There exists a transfinite sequence L_0 such that

(i) $\mathrm{ConsecutiveSet}(A, O) = \mathrm{last}\, L_0$,

(ii) $\mathrm{dom}\, L_0 = \mathrm{succ}\, O$,

(iii) $L_0(\emptyset) = A$,

(iv) for every ordinal number C such that $\mathrm{succ}\, C \in \mathrm{succ}\, O$ holds $L_0(\mathrm{succ}\, C) = L_0(C)^*$, and

(v) for every ordinal number C such that $C \in \mathrm{succ}\, O$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L_0(C) = \bigcup \mathrm{rng}(L_0 \restriction C)$.

One can prove the following three propositions:

(24) $\mathrm{ConsecutiveSet}(A,\emptyset) = A$.

(25) $\mathrm{ConsecutiveSet}(A,\mathrm{succ}\,O) = (\mathrm{ConsecutiveSet}(A,O))^*$.

(26) Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\mathrm{dom}\,T = O$ and for every ordinal number O_1 such that $O_1 \in O$ holds $T(O_1) = \mathrm{ConsecutiveSet}(A,O_1)$. Then $\mathrm{ConsecutiveSet}(A,O) = \bigcup \mathrm{rng}\,T$.

Let us consider A and let us consider O. Observe that $\mathrm{ConsecutiveSet}(A,O)$ is non empty.
One can prove the following proposition

(27) $A \subseteq \mathrm{ConsecutiveSet}(A,O)$.

Let us consider A, L and let d be a bifunction from A into L. A transfinite sequence of elements of $[:A, A,$ the carrier of $L,$ the carrier of $L:]$ is said to be a sequence of quadruples of d if it satisfies the conditions (Def. 14).

(Def. 14)(i) dom it is a cardinal number,

(ii) it is one-to-one, and

(iii) $\mathrm{rng}\,\mathrm{it} = \{\langle x,y,a,b\rangle; x$ ranges over elements of A, y ranges over elements of A, a ranges over elements of L, b ranges over elements of L: $d(x,y) \leq a \sqcup b\}$.

Let us consider A, L, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let us consider O. Let us assume that $O \in \mathrm{dom}\,q$. The functor $\mathrm{Quadr}(q,O)$ yielding an element of $[:\mathrm{ConsecutiveSet}(A,O), \mathrm{ConsecutiveSet}(A,O),$ the carrier of $L,$ the carrier of $L:]$ is defined by:

(Def. 15) $\mathrm{Quadr}(q,O) = q(O)$.

We now state the proposition

(28) Let d be a bifunction from A into L and q be a sequence of quadruples of d. Then $O \in \mathrm{DistEsti}(d)$ if and only if $O \in \mathrm{dom}\,q$.

Let us consider A, L and let z be a set. Let us assume that z is a bifunction from A into L. The functor $\mathrm{BiFun}(z,A,L)$ yielding a bifunction from A into L is defined by:

(Def. 16) $\mathrm{BiFun}(z,A,L) = z$.

Let us consider A, L, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let us consider O. The functor $\mathrm{ConsecutiveDelta}(q,O)$ is defined by the condition (Def. 17).

(Def. 17) There exists a transfinite sequence L_0 such that

(i) $\mathrm{ConsecutiveDelta}(q,O) = \mathrm{last}\,L_0$,

(ii) $\mathrm{dom}\,L_0 = \mathrm{succ}\,O$,

(iii) $L_0(\emptyset) = d$,

(iv) for every ordinal number C such that $\mathrm{succ}\,C \in \mathrm{succ}\,O$ holds $L_0(\mathrm{succ}\,C) = (\mathrm{BiFun}(L_0(C), \mathrm{ConsecutiveSet}(A,C), L))^*_{\mathrm{Quadr}(q,C)}$, and

(v) for every ordinal number C such that $C \in \mathrm{succ}\,O$ and $C \neq \emptyset$ and C is a limit ordinal number holds $L_0(C) = \bigcup \mathrm{rng}(L_0 \restriction C)$.

Next we state several propositions:

(29) For every bifunction d from A into L and for every sequence q of quadruples of d holds $\mathrm{ConsecutiveDelta}(q,\emptyset) = d$.

(30) For every bifunction d from A into L and for every sequence q of quadruples of d holds $\mathrm{ConsecutiveDelta}(q,\mathrm{succ}\,O) = (\mathrm{BiFun}(\mathrm{ConsecutiveDelta}(q,O), \mathrm{ConsecutiveSet}(A,O), L))^*_{\mathrm{Quadr}(q,O)}$.

(31) Let d be a bifunction from A into L and q be a sequence of quadruples of d. Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\mathrm{dom}\, T = O$ and for every ordinal number O_1 such that $O_1 \in O$ holds $T(O_1) = \mathrm{ConsecutiveDelta}(q, O_1)$. Then $\mathrm{ConsecutiveDelta}(q, O) = \bigcup \mathrm{rng}\, T$.

(32) If $O_1 \subseteq O_2$, then $\mathrm{ConsecutiveSet}(A, O_1) \subseteq \mathrm{ConsecutiveSet}(A, O_2)$.

(33) Let d be a bifunction from A into L and q be a sequence of quadruples of d. Then $\mathrm{ConsecutiveDelta}(q, O)$ is a bifunction from $\mathrm{ConsecutiveSet}(A, O)$ into L.

Let us consider A, L, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let us consider O. Then $\mathrm{ConsecutiveDelta}(q, O)$ is a bifunction from $\mathrm{ConsecutiveSet}(A, O)$ into L.

One can prove the following propositions:

(34) For every bifunction d from A into L and for every sequence q of quadruples of d holds $d \subseteq \mathrm{ConsecutiveDelta}(q, O)$.

(35) For every bifunction d from A into L and for every sequence q of quadruples of d such that $O_1 \subseteq O_2$ holds $\mathrm{ConsecutiveDelta}(q, O_1) \subseteq \mathrm{ConsecutiveDelta}(q, O_2)$.

(36) Let d be a bifunction from A into L. Suppose d is zeroed. Let q be a sequence of quadruples of d. Then $\mathrm{ConsecutiveDelta}(q, O)$ is zeroed.

(37) Let d be a bifunction from A into L. Suppose d is symmetric. Let q be a sequence of quadruples of d. Then $\mathrm{ConsecutiveDelta}(q, O)$ is symmetric.

(38) Let d be a bifunction from A into L. Suppose d is symmetric and satisfies triangle inequality. Let q be a sequence of quadruples of d. If $O \subseteq \mathrm{DistEsti}(d)$, then $\mathrm{ConsecutiveDelta}(q, O)$ satisfies triangle inequality.

(39) Let d be a distance function of A, L and q be a sequence of quadruples of d. If $O \subseteq \mathrm{DistEsti}(d)$, then $\mathrm{ConsecutiveDelta}(q, O)$ is a distance function of $\mathrm{ConsecutiveSet}(A, O)$, L.

Let us consider A, L and let d be a bifunction from A into L. The functor $\mathrm{NextSet}(d)$ is defined by:

(Def. 18) $\mathrm{NextSet}(d) = \mathrm{ConsecutiveSet}(A, \mathrm{DistEsti}(d))$.

Let us consider A, L and let d be a bifunction from A into L. Note that $\mathrm{NextSet}(d)$ is non empty.

Let us consider A, L, let d be a bifunction from A into L, and let q be a sequence of quadruples of d. The functor $\mathrm{NextDelta}(q)$ is defined by:

(Def. 19) $\mathrm{NextDelta}(q) = \mathrm{ConsecutiveDelta}(q, \mathrm{DistEsti}(d))$.

Let us consider A, L, let d be a distance function of A, L, and let q be a sequence of quadruples of d. Then $\mathrm{NextDelta}(q)$ is a distance function of $\mathrm{NextSet}(d)$, L.

Let us consider A, L, let d be a distance function of A, L, let A_1 be a non empty set, and let d_1 be a distance function of A_1, L. We say that (A_1, d_1) is extension of (A, d) if and only if:

(Def. 20) There exists a sequence q of quadruples of d such that $A_1 = \mathrm{NextSet}(d)$ and $d_1 = \mathrm{NextDelta}(q)$.

Next we state the proposition

(40) Let d be a distance function of A, L, A_1 be a non empty set, and d_1 be a distance function of A_1, L. Suppose (A_1, d_1) is extension of (A, d). Let x, y be elements of A and a, b be elements of L. Suppose $d(x, y) \leq a \sqcup b$. Then there exist elements z_1, z_2, z_3 of A_1 such that $d_1(x, z_1) = a$ and $d_1(z_2, z_3) = a$ and $d_1(z_1, z_2) = b$ and $d_1(z_3, y) = b$.

Let us consider A, L and let d be a distance function of A, L. A function is called an extension sequence of (A, d) if it satisfies the conditions (Def. 21).

(Def. 21)(i) $\operatorname{dom} \mathrm{it} = \mathbb{N}$,

(ii) $\mathrm{it}(0) = \langle A, d \rangle$, and

(iii) for every natural number n there exists a non empty set A' and there exists a distance function d' of A', L and there exists a non empty set A_1 and there exists a distance function d_1 of A_1, L such that (A_1, d_1) is extension of (A', d') and $\mathrm{it}(n) = \langle A', d' \rangle$ and $\mathrm{it}(n+1) = \langle A_1, d_1 \rangle$.

Next we state two propositions:

(41) Let d be a distance function of A, L, S be an extension sequence of (A, d), and k, l be natural numbers. If $k \leq l$, then $S(k)_{\mathbf{1}} \subseteq S(l)_{\mathbf{1}}$.

(42) Let d be a distance function of A, L, S be an extension sequence of (A, d), and k, l be natural numbers. If $k \leq l$, then $S(k)_{\mathbf{2}} \subseteq S(l)_{\mathbf{2}}$.

Let us consider L. The functor $\delta_0(L)$ yielding a distance function of the carrier of L, L is defined by:

(Def. 22) For all elements x, y of L holds if $x \neq y$, then $(\delta_0(L))(x, y) = x \sqcup y$ and if $x = y$, then $(\delta_0(L))(x, y) = \perp_L$.

We now state several propositions:

(43) $\delta_0(L)$ is onto.

(44) Let S be an extension sequence of (the carrier of L, $\delta_0(L)$) and F_1 be a non empty set. Suppose $F_1 = \bigcup\{S(i)_{\mathbf{1}} : i$ ranges over natural numbers$\}$. Then $\bigcup\{S(i)_{\mathbf{2}} : i$ ranges over natural numbers$\}$ is a distance function of F_1, L.

(45) Let S be an extension sequence of (the carrier of L, $\delta_0(L)$), F_1 be a non empty set, F_2 be a distance function of F_1, L, x, y be elements of F_1, and a, b be elements of L. Suppose $F_1 = \bigcup\{S(i)_{\mathbf{1}} : i$ ranges over natural numbers$\}$ and $F_2 = \bigcup\{S(i)_{\mathbf{2}} : i$ ranges over natural numbers$\}$ and $F_2(x, y) \leq a \sqcup b$. Then there exist elements z_1, z_2, z_3 of F_1 such that $F_2(x, z_1) = a$ and $F_2(z_2, z_3) = a$ and $F_2(z_1, z_2) = b$ and $F_2(z_3, y) = b$.

(46) Let S be an extension sequence of (the carrier of L, $\delta_0(L)$), F_1 be a non empty set, F_2 be a distance function of F_1, L, f be a homomorphism from L to $\mathrm{EqRelPoset}(F_1)$, x, y be elements of F_1, e_1, e_2 be equivalence relations of F_1, and given x, y. Suppose that

(i) $f = \alpha(F_2)$,

(ii) $F_1 = \bigcup\{S(i)_{\mathbf{1}} : i$ ranges over natural numbers$\}$,

(iii) $F_2 = \bigcup\{S(i)_{\mathbf{2}} : i$ ranges over natural numbers$\}$,

(iv) $e_1 \in$ the carrier of $\operatorname{Im} f$,

(v) $e_2 \in$ the carrier of $\operatorname{Im} f$, and

(vi) $\langle x, y \rangle \in e_1 \sqcup e_2$.

Then there exists a non empty finite sequence F of elements of F_1 such that $\operatorname{len} F = 3 + 2$ and x and y are joint by F, e_1 and e_2.

(47) There exists a non empty set A and there exists a homomorphism f from L to $\mathrm{EqRelPoset}(A)$ such that f is one-to-one and the type of $\operatorname{Im} f \leq 3$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[4] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal2.html.

[5] Grzegorz Bancerek. Cartesian product of functions. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/funct_6.html.

[6] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lattice3.html.

[7] Grzegorz Bancerek. Bounds in posets and relational substructures. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/yellow_0.html.

[8] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/waybel_0.html.

[9] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[10] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[11] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[12] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[13] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[14] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[15] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[16] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[17] Robert Milewski. Lattice of congruences in many sorted algebra. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/msualg_5.html.

[18] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[19] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/eqrel_1.html.

[20] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Journal of Formalized Mathematics*, 9, 1997. http://mizar.org/JFM/Vol9/abian.html.

[21] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/domain_1.html.

[22] Andrzej Trybulec. Enumerated sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/enumset1.html.

[23] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[24] Andrzej Trybulec. Tuples, projections and Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/mcart_1.html.

[25] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[26] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/orders_1.html.

[27] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[28] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[29] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

[30] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_2.html.

[31] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/lattices.html.

[32] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/yellow_2.html.

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