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Properties of Caratheodor's Measure

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Summary. The paper contains definitions and basic properties of Caratheodor's measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [10]. The article includes the text being a continuation of the paper [5]. Caratheodor's theorem and some theorems concerning basic properties of Caratheodor's measure are proved. The work is the sixth part of the series of articles concerning the Lebesgue measure theory.

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The articles [11], [8], [13], [12], [14], [6], [7], [1], [9], [2], [3], [4], and [5] provide the notation and terminology for this paper.

We adopt the following convention: x, y, z denote extended real numbers, A, B, X denote sets, and S denotes a σ-field of subsets of X.

We now state several propositions:

(1) If $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$ and $0_{\overline{\mathbb{R}}} \leq z$, then $(x+y)+z = x+(y+z)$.

(2) If $x \neq -\infty$ and $x \neq +\infty$, then $y+x \leq z$ iff $y \leq z-x$.

(3) If $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$, then $x+y = y+x$.

(4) Let S be a non empty family of subsets of X, F, G be functions from $\mathbb{N}$ into S, and A be an element of S. If for every element n of $\mathbb{N}$ holds $G(n) = A \cap F(n)$, then $\bigcup \operatorname{rng} G = A \cap \bigcup \operatorname{rng} F$.

(5) Let S be a non empty family of subsets of X and F, G be functions from $\mathbb{N}$ into S. Suppose $G(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $G(n+1) = F(n+1) \cup G(n)$. Let H be a function from $\mathbb{N}$ into S. Suppose $H(0) = F(0)$ and for every element n of $\mathbb{N}$ holds $H(n+1) = F(n+1) \setminus G(n)$. Then $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} H$.

(6) 2^X is a σ-field of subsets of X.

Let X be a set and let F be a function from $\mathbb{N}$ into 2^X. Then $\operatorname{rng} F$ is a family of subsets of X.

Let X be a set and let A be a family of subsets of X. Then $\bigcup A$ is an element of 2^X.

Let Y be a set, let X, Z be non empty sets, let F be a function from Y into X, and let M be a function from X into Z. Then $M \cdot F$ is a function from Y into Z.

The following three propositions are true:

(7) Let a, b be extended real numbers. Then there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds

(i) if $A = \emptyset$, then $M(A) = a$, and

(ii) if $A \neq \emptyset$, then $M(A) = b$.

(8) There exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds $M(A) = 0_{\overline{\mathbb{R}}}$.

(11)[1] There exists a function M from 2^X into $\overline{\mathbb{R}}$ such that

(i) M is non-negative,

(ii) $M(\emptyset) = 0_{\overline{\mathbb{R}}}$, and

(iii) for all elements A, B of 2^X such that $A \subseteq B$ holds $M(A) \leq M(B)$ and for every function F from $\mathbb{N}$ into 2^X holds $M(\bigcup \operatorname{rng} F) \leq \sum(M \cdot F)$.

Let X be a set. A function from 2^X into $\overline{\mathbb{R}}$ is said to be a Caratheodor's measure on X if it satisfies the conditions (Def. 2).

(Def. 2)[2] (i) It is non-negative,

(ii) $\text{it}(\emptyset) = 0_{\overline{\mathbb{R}}}$, and

(iii) for all elements A, B of 2^X such that $A \subseteq B$ holds $\text{it}(A) \leq \text{it}(B)$ and for every function F from $\mathbb{N}$ into 2^X holds $\text{it}(\bigcup \operatorname{rng} F) \leq \sum(\text{it} \cdot F)$.

In the sequel C denotes a Caratheodor's measure on X.

Let X be a set and let C be a Caratheodor's measure on X. The functor σ-Field(C) yields a non empty family of subsets of X and is defined by the condition (Def. 3).

(Def. 3) Let A be an element of 2^X. Then $A \in \sigma$-Field(C) if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) \leq C(W \cup Z)$.

One can prove the following propositions:

(12) For all elements W, Z of 2^X holds $C(W \cup Z) \leq C(W) + C(Z)$.

(13) For all elements W, Z of 2^X holds $C(Z) + C(W) = C(W) + C(Z)$.

(14) Let A be an element of 2^X. Then $A \in \sigma$-Field(C) if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) = C(W \cup Z)$.

(15) For all elements W, Z of 2^X such that $W \in \sigma$-Field(C) and $Z \in \sigma$-Field(C) and Z misses W holds $C(W \cup Z) = C(W) + C(Z)$.

(16) If $A \in \sigma$-Field(C), then $X \setminus A \in \sigma$-Field(C).

(17) If $A \in \sigma$-Field(C) and $B \in \sigma$-Field(C), then $A \cup B \in \sigma$-Field(C).

(18) If $A \in \sigma$-Field(C) and $B \in \sigma$-Field(C), then $A \cap B \in \sigma$-Field(C).

(19) If $A \in \sigma$-Field(C) and $B \in \sigma$-Field(C), then $A \setminus B \in \sigma$-Field(C).

(20) Let N be a function from $\mathbb{N}$ into S and A be an element of S. Then there exists a function F from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$ holds $F(n) = A \cap N(n)$.

(21) σ-Field(C) is a σ-field of subsets of X.

Let X be a set and let C be a Caratheodor's measure on X. One can check that σ-Field(C) is σ-field of subsets-like, closed for complement operator, and non empty.

Let X be a set, let S be a σ-field of subsets of X, and let A be a subfamily of S. Then $\bigcup A$ is an element of S.

Let X be a set and let C be a Caratheodor's measure on X. The functor σ-Meas(C) yielding a function from σ-Field(C) into $\overline{\mathbb{R}}$ is defined as follows:

(Def. 4) For every element A of 2^X such that $A \in \sigma$-Field(C) holds $(\sigma\text{-Meas}(C))(A) = C(A)$.

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[1] The propositions (9) and (10) have been removed.

[2] The definition (Def. 1) has been removed.

We now state the proposition

(22) σ-Meas(C) is a measure on σ-Field(C).

Let X be a set, let C be a Caratheodor's measure on X, and let A be an element of σ-Field(C). Then $C(A)$ is an extended real number.

One can prove the following proposition

(23) σ-Meas(C) is a σ-measure on σ-Field(C).

Let X be a set and let C be a Caratheodor's measure on X. Then σ-Meas(C) is a σ-measure on σ-Field(C).

One can prove the following two propositions:

(24) For every element A of 2^X such that $C(A) = 0_{\overline{\mathbb{R}}}$ holds $A \in$ σ-Field(C).

(25) σ-Meas(C) is complete on σ-Field(C).

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