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Category of Left Modules

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Summary. We define the category of left modules over an associative ring. The carriers of the modules are included in a universum. The universum is a parameter of the category.

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The articles [10], [5], [13], [14], [2], [3], [12], [6], [4], [11], [9], [7], [8], and [1] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: x, y denote sets, D denotes a non empty set, U_1 denotes a universal class, R denotes a ring, and G, H denote left modules over R.

Let us consider R. A non empty set is called a non empty set of left-modules of R if:

(Def. 1) Every element of it is a strict left module over R.

In the sequel V is a non empty set of left-modules of R.

Let us consider R, V. We see that the element of V is a left module over R.

Let us consider R, V. Note that there exists an element of V which is strict.

Let us consider R. A non empty set is called a non empty set of morphisms of left-modules of R if:

(Def. 2) Every element of it is a strict left module morphism of R.

Let us consider R and let M be a non empty set of morphisms of left-modules of R. We see that the element of M is a left module morphism of R.

Let us consider R and let M be a non empty set of morphisms of left-modules of R. One can verify that there exists an element of M which is strict.

The following proposition is true

(3)[1] For every strict left module morphism f of R holds $\{f\}$ is a non empty set of morphisms of left-modules of R.

Let us consider R, G, H. A non empty set of morphisms of left-modules of R is said to be a non empty set of morphisms of left-modules from G into H if:

(Def. 3) Every element of it is a strict morphism from G to H.

We now state two propositions:

(4) D is a non empty set of morphisms of left-modules from G into H if and only if every element of D is a strict morphism from G to H.

[1] The propositions (1) and (2) have been removed.

(5) Let f be a strict morphism from G to H. Then $\{f\}$ is a non empty set of morphisms of left-modules from G into H.

Let us consider R, G, H. The functor $\mathrm{Morphs}(G,H)$ yields a non empty set of morphisms of left-modules from G into H and is defined by:

(Def. 4) $x \in \mathrm{Morphs}(G,H)$ iff x is a strict morphism from G to H.

Let us consider R, G, H and let M be a non empty set of morphisms of left-modules from G into H. We see that the element of M is a morphism from G to H.

Let us consider x, y, R. The predicate $\mathrm{P}_{\mathrm{ob}}\, x, y, R$ is defined by the condition (Def. 5).

(Def. 5) There exist sets x_1, x_2 such that

(i) $x = \langle x_1, x_2 \rangle$, and

(ii) there exists a strict left module G over R such that $y = G$ and $x_1 =$ the loop structure of G and $x_2 =$ the left multiplication of G.

Next we state two propositions:

(6) For all sets x, y_1, y_2 such that $\mathrm{P}_{\mathrm{ob}}\, x, y_1, R$ and $\mathrm{P}_{\mathrm{ob}}\, x, y_2, R$ holds $y_1 = y_2$.

(7) For every U_1 there exists x such that $x \in \{\langle G, f \rangle : G$ ranges over elements of $\mathrm{GroupObj}(U_1)$, f ranges over elements of $\{\emptyset\}^{[:\text{the carrier of } R,\ \{\emptyset\}:]}\}$ and $\mathrm{P}_{\mathrm{ob}}\, x, {}_R\Theta, R$.

Let us consider U_1, R. The functor $\mathrm{LModObj}(U_1,R)$ yields a set and is defined by the condition (Def. 6).

(Def. 6) Let given y. Then $y \in \mathrm{LModObj}(U_1,R)$ if and only if there exists x such that $x \in \{\langle G, f \rangle : G$ ranges over elements of $\mathrm{GroupObj}(U_1)$, f ranges over elements of $\{\emptyset\}^{[:\text{the carrier of } R,\ \{\emptyset\}:]}\}$ and $\mathrm{P}_{\mathrm{ob}}\, x, y, R$.

One can prove the following proposition

(8) ${}_R\Theta \in \mathrm{LModObj}(U_1,R)$.

Let us consider U_1, R. Note that $\mathrm{LModObj}(U_1,R)$ is non empty.

We now state the proposition

(9) Every element of $\mathrm{LModObj}(U_1,R)$ is a strict left module over R.

Let us consider U_1, R. Then $\mathrm{LModObj}(U_1,R)$ is a non empty set of left-modules of R.

Let us consider R, V. The functor $\mathrm{Morphs}\, V$ yielding a non empty set of morphisms of left-modules of R is defined by:

(Def. 7) For every x holds $x \in \mathrm{Morphs}\, V$ iff there exist strict elements G, H of V such that x is a strict morphism from G to H.

Let us consider R, V and let F be an element of $\mathrm{Morphs}\, V$. The functor $\mathrm{dom}' F$ yielding an element of V is defined by:

(Def. 8) $\mathrm{dom}' F = \mathrm{dom} F$.

The functor $\mathrm{cod}' F$ yields an element of V and is defined as follows:

(Def. 9) $\mathrm{cod}' F = \mathrm{cod} F$.

Let us consider R, V and let G be an element of V. The functor I_G yields a strict element of $\mathrm{Morphs}\, V$ and is defined by:

(Def. 10) $\mathrm{I}_G = \mathrm{I}_G$.

Let us consider R, V. The functor $\mathrm{dom}\, V$ yielding a function from $\mathrm{Morphs}\, V$ into V is defined as follows:

(Def. 11) For every element f of Morphs V holds $(\mathrm{dom}\,V)(f) = \mathrm{dom}' f$.

The functor cod V yields a function from Morphs V into V and is defined by:

(Def. 12) For every element f of Morphs V holds $(\mathrm{cod}\,V)(f) = \mathrm{cod}' f$.

The functor I_V yielding a function from V into Morphs V is defined as follows:

(Def. 13) For every element G of V holds $\mathrm{I}_V(G) = \mathrm{I}_G$.

The following three propositions are true:

(10) Let g, f be elements of Morphs V. Suppose $\mathrm{dom}' g = \mathrm{cod}' f$. Then there exist strict elements G_1, G_2, G_3 of V such that g is a morphism from G_2 to G_3 and f is a morphism from G_1 to G_2.

(11) For all elements g, f of Morphs V such that $\mathrm{dom}' g = \mathrm{cod}' f$ holds $g \cdot f \in$ Morphs V.

(12) For all elements g, f of Morphs V such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $g \cdot f \in$ Morphs V.

Let us consider R, V. The functor comp V yields a partial function from $[:\mathrm{Morphs}\,V, \mathrm{Morphs}\,V:]$ to Morphs V and is defined by the conditions (Def. 14).

(Def. 14)(i) For all elements g, f of Morphs V holds $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\,V$ iff $\mathrm{dom}' g = \mathrm{cod}' f$, and

(ii) for all elements g, f of Morphs V such that $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\,V$ holds $(\mathrm{comp}\,V)(\langle g, f\rangle) = g \cdot f$.

Next we state the proposition

(13) For all elements g, f of Morphs V holds $\langle g, f\rangle \in \mathrm{dom}\,\mathrm{comp}\,V$ iff $\mathrm{dom}\, g = \mathrm{cod}\, f$.

Let us consider U_1, R. The functor $\mathrm{LModCat}(U_1,R)$ yields a strict category structure and is defined as follows:

(Def. 15) $\mathrm{LModCat}(U_1,R) = \langle \mathrm{LModObj}(U_1,R), \mathrm{Morphs}\,\mathrm{LModObj}(U_1,R), \mathrm{dom}\,\mathrm{LModObj}(U_1,R), \mathrm{cod}\,\mathrm{LModObj}(U_1,R), \mathrm{comp}\,\mathrm{LModObj}(U_1,R), \mathrm{I}_{\mathrm{LModObj}(U_1,R)}\rangle$.

One can prove the following propositions:

(14) For all morphisms f, g of $\mathrm{LModCat}(U_1,R)$ holds $\langle g, f\rangle \in$ dom(the composition of $\mathrm{LModCat}(U_1,R)$) iff $\mathrm{dom}\, g = \mathrm{cod}\, f$.

(15) Let f be a morphism of $\mathrm{LModCat}(U_1,R)$, f' be an element of Morphs $\mathrm{LModObj}(U_1,R)$, b be an object of $\mathrm{LModCat}(U_1,R)$, and b' be an element of $\mathrm{LModObj}(U_1,R)$. Then

(i) f is a strict element of Morphs $\mathrm{LModObj}(U_1,R)$,

(ii) f' is a morphism of $\mathrm{LModCat}(U_1,R)$,

(iii) b is a strict element of $\mathrm{LModObj}(U_1,R)$, and

(iv) b' is an object of $\mathrm{LModCat}(U_1,R)$.

(16) For every object b of $\mathrm{LModCat}(U_1,R)$ and for every element b' of $\mathrm{LModObj}(U_1,R)$ such that $b = b'$ holds $\mathrm{id}_b = \mathrm{I}_{b'}$.

(17) For every morphism f of $\mathrm{LModCat}(U_1,R)$ and for every element f' of Morphs $\mathrm{LModObj}(U_1,R)$ such that $f = f'$ holds $\mathrm{dom}\, f = \mathrm{dom}\, f'$ and $\mathrm{cod}\, f = \mathrm{cod}\, f'$.

(18) Let f, g be morphisms of $\mathrm{LModCat}(U_1,R)$ and f', g' be elements of Morphs $\mathrm{LModObj}(U_1,R)$ such that $f = f'$ and $g = g'$. Then

(i) $\mathrm{dom}\, g = \mathrm{cod}\, f$ iff $\mathrm{dom}\, g' = \mathrm{cod}\, f'$,

(ii) $\mathrm{dom}\, g = \mathrm{cod}\, f$ iff $\langle g', f'\rangle \in \mathrm{dom}\,\mathrm{comp}\,\mathrm{LModObj}(U_1,R)$,

(iii) if $\mathrm{dom}\, g = \mathrm{cod}\, f$, then $g \cdot f = g' \cdot f'$,

(iv) $\mathrm{dom}\, f = \mathrm{dom}\, g$ iff $\mathrm{dom}\, f' = \mathrm{dom}\, g'$, and

(v) $\mathrm{cod}\, f = \mathrm{cod}\, g$ iff $\mathrm{cod}\, f' = \mathrm{cod}\, g'$.

Let us consider U_1, R. Note that $\mathrm{LModCat}(U_1,R)$ is category-like.

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