

# Order Sorted Algebras<sup>1</sup>

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**Summary.** Initial notions for order sorted algebras.

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The articles [10], [14], [15], [17], [3], [16], [4], [9], [8], [5], [2], [6], [1], [11], [13], [12], and [7] provide the notation and terminology for this paper.

## 1. PRELIMINARIES

Let  $I$  be a set, let  $f$  be a many sorted set indexed by  $I$ , and let  $p$  be a finite sequence of elements of  $I$ . Observe that  $f \cdot p$  is finite sequence-like.

Let  $S$  be a non empty many sorted signature. A sort symbol of  $S$  is an element of  $S$ .

Let  $S$  be a non empty many sorted signature. An operation symbol of  $S$  is an element of the operation symbols of  $S$ .

Let  $S$  be a non void non empty many sorted signature and let  $o$  be an operation symbol of  $S$ . Then the result sort of  $o$  is an element of  $S$ .

We consider overloaded many sorted signatures as extensions of many sorted signature as systems

$\langle$  a carrier, operation symbols, an overloading, an arity, a result sort  $\rangle$ ,

where the carrier is a set, the operation symbols constitute a set, the overloading is an equivalence relation of the operation symbols, the arity is a function from the operation symbols into the carrier\*, and the result sort is a function from the operation symbols into the carrier.

We consider relation sorted signatures as extensions of many sorted signature and relational structure as systems

$\langle$  a carrier, an internal relation, operation symbols, an arity, a result sort  $\rangle$ ,

where the carrier is a set, the internal relation is a binary relation on the carrier, the operation symbols constitute a set, the arity is a function from the operation symbols into the carrier\*, and the result sort is a function from the operation symbols into the carrier.

We consider overloaded relation sorted signatures as extensions of overloaded many sorted signature and relation sorted signature as systems

$\langle$  a carrier, an internal relation, operation symbols, an overloading, an arity, a result sort  $\rangle$ ,

where the carrier is a set, the internal relation is a binary relation on the carrier, the operation symbols constitute a set, the overloading is an equivalence relation of the operation symbols, the arity is a function from the operation symbols into the carrier\*, and the result sort is a function from the operation symbols into the carrier.

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For simplicity, we adopt the following convention:  $A, O$  are non empty sets,  $R$  is an order in  $A$ ,  $O_1$  is an equivalence relation of  $O$ ,  $f$  is a function from  $O$  into  $A^*$ , and  $g$  is a function from  $O$  into  $A$ . One can prove the following proposition

- (1)  $\langle A, R, O, O_1, f, g \rangle$  is non empty, non void, reflexive, transitive, and antisymmetric.

Let us consider  $A, R, O, O_1, f, g$ . One can check that  $\langle A, R, O, O_1, f, g \rangle$  is strict, non empty, reflexive, transitive, and antisymmetric.

## 2. THE NOTIONS: ORDER-SORTED, DISCERNABLE, OP-DISCRETE

In the sequel  $S$  denotes an overloaded relation sorted signature.

Let us consider  $S$ . We say that  $S$  is order-sorted if and only if:

- (Def. 2)<sup>1</sup>  $S$  is reflexive, transitive, and antisymmetric.

Let us observe that every overloaded relation sorted signature which is order-sorted is also reflexive, transitive, and antisymmetric and there exists an overloaded relation sorted signature which is strict, non empty, non void, and order-sorted.

One can verify that there exists an overloaded many sorted signature which is non empty and non void.

Let  $S$  be a non empty non void overloaded many sorted signature and let  $x, y$  be operation symbols of  $S$ . The predicate  $x \cong y$  is defined as follows:

- (Def. 3)  $\langle x, y \rangle \in$  the overloading of  $S$ .

Let us notice that the predicate  $x \cong y$  is reflexive and symmetric.

The following proposition is true

- (2) Let  $S$  be a non empty non void overloaded many sorted signature and  $o, o_1, o_2$  be operation symbols of  $S$ . If  $o \cong o_1$  and  $o_1 \cong o_2$ , then  $o \cong o_2$ .

Let  $S$  be a non empty non void overloaded many sorted signature. We say that  $S$  is discernable if and only if the condition (Def. 4) is satisfied.

- (Def. 4) Let  $x, y$  be operation symbols of  $S$ . Suppose  $x \cong y$  and  $\text{Arity}(x) = \text{Arity}(y)$  and the result sort of  $x =$  the result sort of  $y$ . Then  $x = y$ .

We say that  $S$  is op-discrete if and only if:

- (Def. 5) The overloading of  $S = \text{id}_{\text{the operation symbols of } S}$ .

One can prove the following propositions:

- (3) Let  $S$  be a non empty non void overloaded many sorted signature. Then  $S$  is op-discrete if and only if for all operation symbols  $x, y$  of  $S$  such that  $x \cong y$  holds  $x = y$ .
- (4) For every non empty non void overloaded many sorted signature  $S$  such that  $S$  is op-discrete holds  $S$  is discernable.

## 3. ORDER SORTED SIGNATURE

In the sequel  $S_0$  is a non empty non void many sorted signature.

Let us consider  $S_0$ . The functor  $\text{OSSign}S_0$  yielding a strict non empty non void order-sorted overloaded relation sorted signature is defined by the conditions (Def. 6).

<sup>1</sup> The definition (Def. 1) has been removed.

- (Def. 6)(i) The carrier of  $S_0 =$  the carrier of  $\text{OSSign } S_0$ ,
- (ii)  $\text{id}_{\text{the carrier of } S_0} =$  the internal relation of  $\text{OSSign } S_0$ ,
  - (iii) the operation symbols of  $S_0 =$  the operation symbols of  $\text{OSSign } S_0$ ,
  - (iv)  $\text{id}_{\text{the operation symbols of } S_0} =$  the overloading of  $\text{OSSign } S_0$ ,
  - (v) the arity of  $S_0 =$  the arity of  $\text{OSSign } S_0$ , and
  - (vi) the result sort of  $S_0 =$  the result sort of  $\text{OSSign } S_0$ .

Next we state the proposition

- (5)  $\text{OSSign } S_0$  is discrete and op-discrete.

Let us note that there exists a strict non empty non void order-sorted overloaded relation sorted signature which is discrete, op-discrete, and discernable.

Let us mention that every non empty non void overloaded relation sorted signature which is op-discrete is also discernable.

Let us consider  $S_0$ . Observe that  $\text{OSSign } S_0$  is discrete and op-discrete.

An order sorted signature is a discernable non empty non void order-sorted overloaded relation sorted signature.

We adopt the following rules:  $S$  denotes a non empty poset,  $s_1, s_2$  denote elements of  $S$ , and  $w_1, w_2$  denote elements of  $(\text{the carrier of } S)^*$ .

Let us consider  $S$  and let  $w_1, w_2$  be elements of  $(\text{the carrier of } S)^*$ . The predicate  $w_1 \leq w_2$  is defined by:

- (Def. 7)  $\text{len } w_1 = \text{len } w_2$  and for every set  $i$  such that  $i \in \text{dom } w_1$  and for all  $s_1, s_2$  such that  $s_1 = w_1(i)$  and  $s_2 = w_2(i)$  holds  $s_1 \leq s_2$ .

Let us note that the predicate  $w_1 \leq w_2$  is reflexive.

We now state two propositions:

- (6) For all elements  $w_1, w_2$  of  $(\text{the carrier of } S)^*$  such that  $w_1 \leq w_2$  and  $w_2 \leq w_1$  holds  $w_1 = w_2$ .
- (7) If  $S$  is discrete and  $w_1 \leq w_2$ , then  $w_1 = w_2$ .

We adopt the following convention:  $S$  is an order sorted signature,  $o, o_1, o_2$  are operation symbols of  $S$ , and  $w_1$  is an element of  $(\text{the carrier of } S)^*$ .

The following proposition is true

- (8) If  $S$  is discrete and  $o_1 \cong o_2$  and  $\text{Arity}(o_1) \leq \text{Arity}(o_2)$  and the result sort of  $o_1 \leq$  the result sort of  $o_2$ , then  $o_1 = o_2$ .

Let us consider  $S$  and let us consider  $o$ . We say that  $o$  is monotone if and only if:

- (Def. 8) For every  $o_2$  such that  $o \cong o_2$  and  $\text{Arity}(o) \leq \text{Arity}(o_2)$  holds the result sort of  $o \leq$  the result sort of  $o_2$ .

Let us consider  $S$ . We say that  $S$  is monotone if and only if:

- (Def. 9) Every operation symbol of  $S$  is monotone.

Next we state the proposition

- (9) If  $S$  is op-discrete, then  $S$  is monotone.

One can verify that there exists an order sorted signature which is monotone.

Let  $S$  be a monotone order sorted signature. One can verify that there exists an operation symbol of  $S$  which is monotone.

Let  $S$  be a monotone order sorted signature. Note that every operation symbol of  $S$  is monotone.

One can check that every order sorted signature which is op-discrete is also monotone.

One can prove the following proposition

(10) If  $S$  is monotone and  $\text{Arity}(o_1) = \emptyset$  and  $o_1 \cong o_2$  and  $\text{Arity}(o_2) = \emptyset$ , then  $o_1 = o_2$ .

Let us consider  $S, o, o_1, w_1$ . We say that  $o_1$  has least args for  $o, w_1$  if and only if:

(Def. 10)  $o \cong o_1$  and  $w_1 \leq \text{Arity}(o_1)$  and for every  $o_2$  such that  $o \cong o_2$  and  $w_1 \leq \text{Arity}(o_2)$  holds  $\text{Arity}(o_1) \leq \text{Arity}(o_2)$ .

We say that  $o_1$  has least sort for  $o, w_1$  if and only if:

(Def. 11)  $o \cong o_1$  and  $w_1 \leq \text{Arity}(o_1)$  and for every  $o_2$  such that  $o \cong o_2$  and  $w_1 \leq \text{Arity}(o_2)$  holds the result sort of  $o_1 \leq$  the result sort of  $o_2$ .

Let us consider  $S, o, o_1, w_1$ . We say that  $o_1$  has least rank for  $o, w_1$  if and only if:

(Def. 12)  $o_1$  has least args for  $o, w_1$  and least sort for  $o, w_1$ .

Let us consider  $S, o$ . We say that  $o$  is regular if and only if:

(Def. 13)  $o$  is monotone and for every  $w_1$  such that  $w_1 \leq \text{Arity}(o)$  holds there exists  $o_1$  which has least args for  $o, w_1$ .

Let  $S_1$  be a monotone order sorted signature. We say that  $S_1$  is regular if and only if:

(Def. 14) Every operation symbol of  $S_1$  is regular.

In the sequel  $S_1$  is a monotone order sorted signature,  $o, o_1$  are operation symbols of  $S_1$ , and  $w_1$  is an element of (the carrier of  $S_1$ )\*.

One can prove the following propositions:

(11)  $S_1$  is regular iff for all  $o, w_1$  such that  $w_1 \leq \text{Arity}(o)$  holds there exists  $o_1$  which has least rank for  $o, w_1$ .

(12) For every monotone order sorted signature  $S_1$  such that  $S_1$  is op-discrete holds  $S_1$  is regular.

Let us observe that there exists a monotone order sorted signature which is regular.

Let us mention that every monotone order sorted signature which is op-discrete is also regular.

Let  $S_2$  be a regular monotone order sorted signature. Observe that every operation symbol of  $S_2$  is regular.

We adopt the following convention:  $S_2$  denotes a regular monotone order sorted signature,  $o, o_3, o_4$  denote operation symbols of  $S_2$ , and  $w_1$  denotes an element of (the carrier of  $S_2$ )\*.

One can prove the following proposition

(13) If  $w_1 \leq \text{Arity}(o)$  and  $o_3$  has least args for  $o, w_1$  and  $o_4$  has least args for  $o, w_1$ , then  $o_3 = o_4$ .

Let us consider  $S_2, o, w_1$ . Let us assume that  $w_1 \leq \text{Arity}(o)$ . The functor  $\text{LBound}(o, w_1)$  yields an operation symbol of  $S_2$  and is defined as follows:

(Def. 15)  $\text{LBound}(o, w_1)$  has least args for  $o, w_1$ .

The following proposition is true

(14) For every  $w_1$  such that  $w_1 \leq \text{Arity}(o)$  holds  $\text{LBound}(o, w_1)$  has least rank for  $o, w_1$ .

In the sequel  $R$  is a non empty poset and  $z$  is a non empty set.

Let us consider  $R, z$ . The functor  $\text{ConstOSSet}(R, z)$  yielding a many sorted set indexed by the carrier of  $R$  is defined as follows:

(Def. 16)  $\text{ConstOSSet}(R, z) = (\text{the carrier of } R) \mapsto z$ .

Next we state the proposition

(15)  $\text{ConstOSSet}(R, z)$  is non-empty and for all elements  $s_1, s_2$  of  $R$  such that  $s_1 \leq s_2$  holds  $(\text{ConstOSSet}(R, z))(s_1) \subseteq (\text{ConstOSSet}(R, z))(s_2)$ .

Let us consider  $R$  and let  $M$  be a many sorted set indexed by  $R$ . We say that  $M$  is order-sorted if and only if:

(Def. 18)<sup>2</sup> For all elements  $s_1, s_2$  of  $R$  such that  $s_1 \leq s_2$  holds  $M(s_1) \subseteq M(s_2)$ .

One can prove the following proposition

(16)  $\text{ConstOSSet}(R, z)$  is order-sorted.

Let us consider  $R$ . One can check that there exists a many sorted set indexed by  $R$  which is order-sorted.

Let us consider  $R, z$ . Then  $\text{ConstOSSet}(R, z)$  is an order-sorted many sorted set indexed by  $R$ .

Let  $R$  be a non empty poset. An order sorted set of  $R$  is an order-sorted many sorted set indexed by  $R$ .

Let  $R$  be a non empty poset. One can check that there exists an order sorted set of  $R$  which is non-empty.

We adopt the following convention:  $s_1, s_2$  denote sort symbols of  $S$ ,  $o, o_1, o_2, o_3$  denote operation symbols of  $S$ , and  $w_1, w_2$  denote elements of (the carrier of  $S$ )<sup>\*</sup>.

Let us consider  $S$  and let  $M$  be an algebra over  $S$ . We say that  $M$  is order-sorted if and only if:

(Def. 19) For all  $s_1, s_2$  such that  $s_1 \leq s_2$  holds (the sorts of  $M$ )( $s_1$ )  $\subseteq$  (the sorts of  $M$ )( $s_2$ ).

One can prove the following proposition

(17) For every algebra  $M$  over  $S$  holds  $M$  is order-sorted iff the sorts of  $M$  are an order sorted set of  $S$ .

In the sequel  $C_1$  is a many sorted function from  $(\text{ConstOSSet}(S, z))^{\#}$  · the arity of  $S$  into  $\text{ConstOSSet}(S, z)$  · the result sort of  $S$ .

Let us consider  $S, z, C_1$ . The functor  $\text{ConstOSA}(S, z, C_1)$  yields a strict non-empty algebra over  $S$  and is defined as follows:

(Def. 20) The sorts of  $\text{ConstOSA}(S, z, C_1) = \text{ConstOSSet}(S, z)$  and the characteristics of  $\text{ConstOSA}(S, z, C_1) = C_1$ .

One can prove the following proposition

(18)  $\text{ConstOSA}(S, z, C_1)$  is order-sorted.

Let us consider  $S$ . Observe that there exists an algebra over  $S$  which is strict, non-empty, and order-sorted.

Let us consider  $S, z, C_1$ . Observe that  $\text{ConstOSA}(S, z, C_1)$  is order-sorted.

Let us consider  $S$ . An order sorted algebra of  $S$  is an order-sorted algebra over  $S$ .

The following proposition is true

(19) For every discrete order sorted signature  $S$  holds every algebra over  $S$  is order-sorted.

Let  $S$  be a discrete order sorted signature. One can verify that every algebra over  $S$  is order-sorted.

In the sequel  $A$  denotes an order sorted algebra of  $S$ .

One can prove the following proposition

(20) If  $w_1 \leq w_2$ , then (the sorts of  $A$ )<sup>#</sup>( $w_1$ )  $\subseteq$  (the sorts of  $A$ )<sup>#</sup>( $w_2$ ).

In the sequel  $M$  is an algebra over  $S_0$ .

Let us consider  $S_0, M$ . The functor  $\text{OSAlg}M$  yielding a strict order sorted algebra of  $\text{OSSign}S_0$  is defined by:

(Def. 21) The sorts of  $\text{OSAlg}M =$  the sorts of  $M$  and the characteristics of  $\text{OSAlg}M =$  the characteristics of  $M$ .

<sup>2</sup> The definition (Def. 17) has been removed.

In the sequel  $A$  denotes an order sorted algebra of  $S$ .  
The following proposition is true

- (21) For all elements  $w_1, w_2, w_3$  of (the carrier of  $S$ )\* such that  $w_1 \leq w_2$  and  $w_2 \leq w_3$  holds  $w_1 \leq w_3$ .

Let us consider  $S, o_1, o_2$ . The predicate  $o_1 \leq o_2$  is defined as follows:

- (Def. 22)  $o_1 \cong o_2$  and  $\text{Arity}(o_1) \leq \text{Arity}(o_2)$  and the result sort of  $o_1 \leq$  the result sort of  $o_2$ .

Let us note that the predicate  $o_1 \leq o_2$  is reflexive.

One can prove the following propositions:

- (22) If  $o_1 \leq o_2$  and  $o_2 \leq o_1$ , then  $o_1 = o_2$ .  
(23) If  $o_1 \leq o_2$  and  $o_2 \leq o_3$ , then  $o_1 \leq o_3$ .  
(24) If the result sort of  $o_1 \leq$  the result sort of  $o_2$ , then  $\text{Result}(o_1, A) \subseteq \text{Result}(o_2, A)$ .  
(25) If  $\text{Arity}(o_1) \leq \text{Arity}(o_2)$ , then  $\text{Args}(o_1, A) \subseteq \text{Args}(o_2, A)$ .  
(26) If  $o_1 \leq o_2$ , then  $\text{Args}(o_1, A) \subseteq \text{Args}(o_2, A)$  and  $\text{Result}(o_1, A) \subseteq \text{Result}(o_2, A)$ .

Let us consider  $S, A$ . We say that  $A$  is monotone if and only if:

- (Def. 23) For all  $o_1, o_2$  such that  $o_1 \leq o_2$  holds  $\text{Den}(o_2, A) \upharpoonright \text{Args}(o_1, A) = \text{Den}(o_1, A)$ .

One can prove the following two propositions:

- (27) Let  $A$  be a non-empty order sorted algebra of  $S$ . Then  $A$  is monotone if and only if for all  $o_1, o_2$  such that  $o_1 \leq o_2$  holds  $\text{Den}(o_1, A) \subseteq \text{Den}(o_2, A)$ .  
(28) If  $S$  is discrete and op-discrete, then  $A$  is monotone.

Let us consider  $S, z$  and let  $z_1$  be an element of  $z$ . The functor  $\text{TrivialOSA}(S, z, z_1)$  yielding a strict order sorted algebra of  $S$  is defined by:

- (Def. 24) The sorts of  $\text{TrivialOSA}(S, z, z_1) = \text{ConstOSSet}(S, z)$  and for every  $o$  holds  $\text{Den}(o, \text{TrivialOSA}(S, z, z_1)) = \text{Args}(o, \text{TrivialOSA}(S, z, z_1)) \mapsto z_1$ .

We now state the proposition

- (29) For every element  $z_1$  of  $z$  holds  $\text{TrivialOSA}(S, z, z_1)$  is non-empty and  $\text{TrivialOSA}(S, z, z_1)$  is monotone.

Let us consider  $S$ . One can verify that there exists an order sorted algebra of  $S$  which is monotone, strict, and non-empty.

Let us consider  $S, z$  and let  $z_1$  be an element of  $z$ . One can verify that  $\text{TrivialOSA}(S, z, z_1)$  is monotone and non-empty.

In the sequel  $o_5, o_6$  denote operation symbols of  $S$ .

Let us consider  $S$ . The functor  $\text{OperNames}S$  yielding a non empty family of subsets of the operation symbols of  $S$  is defined by:

- (Def. 25)  $\text{OperNames}S = \text{Classes}$ (the overloading of  $S$ ).

Let us consider  $S$ . One can verify that every element of  $\text{OperNames}S$  is non empty.

Let us consider  $S$ . An  $\text{OperName}$  of  $S$  is an element of  $\text{OperNames}S$ .

Let us consider  $S, o_5$ . The functor  $\text{Name}o_5$  yields an  $\text{OperName}$  of  $S$  and is defined by:

- (Def. 26)  $\text{Name}o_5 = [o_5]_{\text{the overloading of } S}$ .

We now state three propositions:

- (30)  $o_5 \cong o_6$  iff  $o_6 \in [o_5]_{\text{the overloading of } S}$ .
- (31)  $o_5 \cong o_6$  iff  $\text{Name } o_5 = \text{Name } o_6$ .
- (32) For every set  $X$  holds  $X$  is an OperName of  $S$  iff there exists  $o_5$  such that  $X = \text{Name } o_5$ .

Let us consider  $S$  and let  $o$  be an OperName of  $S$ . We see that the element of  $o$  is an operation symbol of  $S$ .

One can prove the following propositions:

- (33) For every OperName  $o_8$  of  $S$  and for every operation symbol  $o_7$  of  $S$  holds  $o_7$  is an element of  $o_8$  iff  $\text{Name } o_7 = o_8$ .
- (34) Let  $S_2$  be a regular monotone order sorted signature,  $o_5, o_6$  be operation symbols of  $S_2$ , and  $w$  be an element of  $(\text{the carrier of } S_2)^*$ . If  $o_5 \cong o_6$  and  $\text{len Arity}(o_5) = \text{len Arity}(o_6)$  and  $w \leq \text{Arity}(o_5)$  and  $w \leq \text{Arity}(o_6)$ , then  $\text{LBound}(o_5, w) = \text{LBound}(o_6, w)$ .

Let  $S_2$  be a regular monotone order sorted signature, let  $o_8$  be an OperName of  $S_2$ , and let  $w$  be an element of  $(\text{the carrier of } S_2)^*$ . Let us assume that there exists an element  $o_7$  of  $o_8$  such that  $w \leq \text{Arity}(o_7)$ . The functor  $\text{LBound}(o_8, w)$  yielding an element of  $o_8$  is defined as follows:

(Def. 27) For every element  $o_7$  of  $o_8$  such that  $w \leq \text{Arity}(o_7)$  holds  $\text{LBound}(o_8, w) = \text{LBound}(o_7, w)$ .

Next we state the proposition

- (35) Let  $S$  be a regular monotone order sorted signature,  $o$  be an operation symbol of  $S$ , and  $w_1$  be an element of  $(\text{the carrier of } S)^*$ . If  $w_1 \leq \text{Arity}(o)$ , then  $\text{LBound}(o, w_1) \leq o$ .

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