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Partial Functions

Czesław Byliński
Warsaw University
Białystok

Summary. In the article we define partial functions. We also define the following notions related to partial functions and functions themselves: the empty function, the restriction of a function to a partial function from a set into a set, the set of all partial functions from a set into a set, the total functions, the relation of tolerance of two functions and the set of all total functions which are tolerated by a partial function. Some simple propositions related to the introduced notions are proved. In the beginning of this article we prove some auxiliary theorems and schemes related to the articles: [1] and [2].

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The articles [4], [3], [5], [6], [7], and [1] provide the notation and terminology for this paper.

1. MAIN PART

In this paper x, y, y_1, y_2, z, z_1, z_2, P, Q, X, X_1, X_2, Y, Y_1, Y_2, V, Z are sets.

The following propositions are true:

(1) If $P \subseteq [: X_1, Y_1 :]$ and $Q \subseteq [: X_2, Y_2 :]$, then $P \cup Q \subseteq [: X_1 \cup X_2, Y_1 \cup Y_2 :]$.

(2) For all functions f, g such that for every x such that $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ holds $f(x) = g(x)$ there exists a function h such that $f \cup g = h$.

(3) For all functions f, g, h such that $f \cup g = h$ and for every x such that $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$ holds $f(x) = g(x)$.

The scheme *LambdaC* deals with a set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

> There exists a function f such that $\mathrm{dom}\, f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ and if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$

for all values of the parameters.

Let us note that there exists a function which is empty.

The following proposition is true

(10)[1] $\mathrm{rng}\, \emptyset = \emptyset$.

Let us consider X, Y. Note that there exists a relation between X and Y which is function-like.

Let us consider X, Y. A partial function from X to Y is a function-like relation between X and Y.

One can prove the following propositions:

[1] The propositions (4)–(9) have been removed.

(24)[2] Every function f is a partial function from dom f to rng f.

(25) For every function f such that rng $f \subseteq Y$ holds f is a partial function from dom f to Y.

(26) For every partial function f from X to Y such that $y \in$ rng f there exists an element x of X such that $x \in$ dom f and $y = f(x)$.

(27) For every partial function f from X to Y such that $x \in$ dom f holds $f(x) \in Y$.

(28) For every partial function f from X to Y such that dom $f \subseteq Z$ holds f is a partial function from Z to Y.

(29) For every partial function f from X to Y such that rng $f \subseteq Z$ holds f is a partial function from X to Z.

(30) For every partial function f from X to Y such that $X \subseteq Z$ holds f is a partial function from Z to Y.

(31) For every partial function f from X to Y such that $Y \subseteq Z$ holds f is a partial function from X to Z.

(32) For every partial function f from X_1 to Y_1 such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ holds f is a partial function from X_2 to Y_2.

(33) Let f be a function and g be a partial function from X to Y. If $f \subseteq g$, then f is a partial function from X to Y.

(34) Let f_1, f_2 be partial functions from X to Y. Suppose dom $f_1 =$ dom f_2 and for every element x of X such that $x \in$ dom f_1 holds $f_1(x) = f_2(x)$. Then $f_1 = f_2$.

(35) Let f_1, f_2 be partial functions from $[:X, Y:]$ to Z. If dom $f_1 =$ dom f_2 and for all x, y such that $\langle x, y\rangle \in$ dom f_1 holds $f_1(\langle x, y\rangle) = f_2(\langle x, y\rangle)$, then $f_1 = f_2$.

Now we present four schemes. The scheme *PartFuncEx* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every x holds $x \in$ dom f iff $x \in \mathcal{A}$ and there exists y such that $\mathcal{P}[x,y]$ and for every x such that $x \in$ dom f holds $\mathcal{P}[x, f(x)]$

provided the parameters meet the following requirements:

- For all x, y such that $x \in \mathcal{A}$ and $\mathcal{P}[x,y]$ holds $y \in \mathcal{B}$, and
- For all x, y_1, y_2 such that $x \in \mathcal{A}$ and $\mathcal{P}[x,y_1]$ and $\mathcal{P}[x,y_2]$ holds $y_1 = y_2$.

The scheme *LambdaR* deals with sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

There exists a partial function f from $\mathcal{A}$ to $\mathcal{B}$ such that for every x holds $x \in$ dom f iff $x \in \mathcal{A}$ and $\mathcal{P}[x]$ and for every x such that $x \in$ dom f holds $f(x) = \mathcal{F}(x)$

provided the parameters have the following property:

- For every x such that $\mathcal{P}[x]$ holds $\mathcal{F}(x) \in \mathcal{B}$.

The scheme *PartFuncEx2* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a ternary predicate $\mathcal{P}$, and states that:

There exists a partial function f from $[:\mathcal{A}, \mathcal{B}:]$ to $\mathcal{C}$ such that

(i) for all x, y holds $\langle x, y\rangle \in$ dom f iff $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and there exists z such that $\mathcal{P}[x,y,z]$, and

(ii) for all x, y such that $\langle x, y\rangle \in$ dom f holds $\mathcal{P}[x,y, f(\langle x, y\rangle)]$

provided the parameters meet the following conditions:

- For all x, y, z such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x,y,z]$ holds $z \in \mathcal{C}$, and
- For all x, y, z_1, z_2 such that $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x,y,z_1]$ and $\mathcal{P}[x,y,z_2]$ holds $z_1 = z_2$.

The scheme *LambdaR2* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, a binary functor $\mathcal{F}$ yielding a set, and a binary predicate $\mathcal{P}$, and states that:

[2] The propositions (11)–(23) have been removed.

There exists a partial function f from $[:\mathcal{A}, \mathcal{B}:]$ to $\mathcal{C}$ such that for all x, y holds $\langle x, y\rangle \in \operatorname{dom} f$ iff $x \in \mathcal{A}$ and $y \in \mathcal{B}$ and $\mathcal{P}[x,y]$ and for all x, y such that $\langle x, y\rangle \in \operatorname{dom} f$ holds $f(\langle x, y\rangle) = \mathcal{F}(x,y)$

provided the following requirement is met:

- For all x, y such that $\mathcal{P}[x,y]$ holds $\mathcal{F}(x,y) \in \mathcal{C}$.

Let us consider X, Y, V, Z, let f be a partial function from X to Y, and let g be a partial function from V to Z. Then $g \cdot f$ is a partial function from X to Z.

Next we state several propositions:

(36) For every partial function f from X to Y holds $f \cdot \mathrm{id}_X = f$.

(37) For every partial function f from X to Y holds $\mathrm{id}_Y \cdot f = f$.

(38) Let f be a partial function from X to Y. Suppose that for all elements x_1, x_2 of X such that $x_1 \in \operatorname{dom} f$ and $x_2 \in \operatorname{dom} f$ and $f(x_1) = f(x_2)$ holds $x_1 = x_2$. Then f is one-to-one.

(39) For every partial function f from X to Y such that f is one-to-one holds f^{-1} is a partial function from Y to X.

(43)[3] For every partial function f from X to Y holds $f \restriction Z$ is a partial function from Z to Y.

(44) For every partial function f from X to Y holds $f \restriction Z$ is a partial function from X to Y.

Let us consider X, Y, let f be a partial function from X to Y, and let Z be a set. Then $f \restriction Z$ is a partial function from X to Y.

Next we state a number of propositions:

(45) For every partial function f from X to Y holds $Z \upharpoonright f$ is a partial function from X to Z.

(46) For every partial function f from X to Y holds $Z \upharpoonright f$ is a partial function from X to Y.

(47) For every function f holds $Y \upharpoonright f \restriction X$ is a partial function from X to Y.

(49)[4] For every partial function f from X to Y such that $y \in f^\circ X$ there exists an element x of X such that $x \in \operatorname{dom} f$ and $y = f(x)$.

(51)[5] For every partial function f from X to Y holds $f^\circ X = \operatorname{rng} f$.

(53)[6] For every partial function f from X to Y holds $f^{-1}(Y) = \operatorname{dom} f$.

(54) For every partial function f from $\emptyset$ to Y holds $\operatorname{dom} f = \emptyset$ and $\operatorname{rng} f = \emptyset$.

(55) For every function f such that $\operatorname{dom} f = \emptyset$ holds f is a partial function from X to Y.

(56) $\emptyset$ is a partial function from X to Y.

(57) For every partial function f from $\emptyset$ to Y holds $f = \emptyset$.

(58) For every partial function f_1 from $\emptyset$ to Y_1 and for every partial function f_2 from $\emptyset$ to Y_2 holds $f_1 = f_2$.

(59) Every partial function from $\emptyset$ to Y is one-to-one.

(60) For every partial function f from $\emptyset$ to Y holds $f^\circ P = \emptyset$.

(61) For every partial function f from $\emptyset$ to Y holds $f^{-1}(Q) = \emptyset$.

(62) For every partial function f from X to $\emptyset$ holds $\operatorname{dom} f = \emptyset$ and $\operatorname{rng} f = \emptyset$.

[3] The propositions (40)–(42) have been removed.
[4] The proposition (48) has been removed.
[5] The proposition (50) has been removed.
[6] The proposition (52) has been removed.

(63) For every function f such that $\operatorname{rng} f = \emptyset$ holds f is a partial function from X to Y.

(64) For every partial function f from X to $\emptyset$ holds $f = \emptyset$.

(65) For every partial function f_1 from X_1 to $\emptyset$ and for every partial function f_2 from X_2 to $\emptyset$ holds $f_1 = f_2$.

(66) Every partial function from X to $\emptyset$ is one-to-one.

(67) For every partial function f from X to $\emptyset$ holds $f^{\circ}P = \emptyset$.

(68) For every partial function f from X to $\emptyset$ holds $f^{-1}(Q) = \emptyset$.

(69) For every partial function f from $\{x\}$ to Y holds $\operatorname{rng} f \subseteq \{f(x)\}$.

(70) Every partial function from $\{x\}$ to Y is one-to-one.

(71) For every partial function f from $\{x\}$ to Y holds $f^{\circ}P \subseteq \{f(x)\}$.

(72) For every function f such that $\operatorname{dom} f = \{x\}$ and $x \in X$ and $f(x) \in Y$ holds f is a partial function from X to Y.

(73) For every partial function f from X to $\{y\}$ such that $x \in \operatorname{dom} f$ holds $f(x) = y$.

(74) For all partial functions f_1, f_2 from X to $\{y\}$ such that $\operatorname{dom} f_1 = \operatorname{dom} f_2$ holds $f_1 = f_2$.

Let f be a function and let X, Y be sets. The functor $f_{\restriction X \dot{\to} Y}$ yields a partial function from X to Y and is defined as follows:

(Def. 3)[7] $f_{\restriction X \dot{\to} Y} = Y \restriction f \restriction X$.

The following propositions are true:

(76)[8] For every function f holds $f_{\restriction X \dot{\to} Y} \subseteq f$.

(77) For every function f holds $\operatorname{dom}(f_{\restriction X \dot{\to} Y}) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(f_{\restriction X \dot{\to} Y}) \subseteq \operatorname{rng} f$.

(78) For every function f holds $x \in \operatorname{dom}(f_{\restriction X \dot{\to} Y})$ iff $x \in \operatorname{dom} f$ and $x \in X$ and $f(x) \in Y$.

(79) For every function f such that $x \in \operatorname{dom} f$ and $x \in X$ and $f(x) \in Y$ holds $f_{\restriction X \dot{\to} Y}(x) = f(x)$.

(80) For every function f such that $x \in \operatorname{dom}(f_{\restriction X \dot{\to} Y})$ holds $f_{\restriction X \dot{\to} Y}(x) = f(x)$.

(81) For all functions f, g such that $f \subseteq g$ holds $f_{\restriction X \dot{\to} Y} \subseteq g_{\restriction X \dot{\to} Y}$.

(82) For every function f such that $Z \subseteq X$ holds $f_{\restriction Z \dot{\to} Y} \subseteq f_{\restriction X \dot{\to} Y}$.

(83) For every function f such that $Z \subseteq Y$ holds $f_{\restriction X \dot{\to} Z} \subseteq f_{\restriction X \dot{\to} Y}$.

(84) For every function f such that $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ holds $f_{\restriction X_1 \dot{\to} Y_1} \subseteq f_{\restriction X_2 \dot{\to} Y_2}$.

(85) For every function f such that $\operatorname{dom} f \subseteq X$ and $\operatorname{rng} f \subseteq Y$ holds $f = f_{\restriction X \dot{\to} Y}$.

(86) For every function f holds $f = f_{\restriction \operatorname{dom} f \dot{\to} \operatorname{rng} f}$.

(87) For every partial function f from X to Y holds $f_{\restriction X \dot{\to} Y} = f$.

(91)[9] $\emptyset_{\restriction X \dot{\to} Y} = \emptyset$.

(92) For all functions f, g holds $g_{\restriction Y \dot{\to} Z} \cdot f_{\restriction X \dot{\to} Y} \subseteq (g \cdot f)_{\restriction X \dot{\to} Z}$.

(93) For all functions f, g such that $\operatorname{rng} f \cap \operatorname{dom} g \subseteq Y$ holds $g_{\restriction Y \dot{\to} Z} \cdot f_{\restriction X \dot{\to} Y} = (g \cdot f)_{\restriction X \dot{\to} Z}$.

[7] The definitions (Def. 1) and (Def. 2) have been removed.
[8] The proposition (75) has been removed.
[9] The propositions (88)–(90) have been removed.

(94) For every function f such that f is one-to-one holds $f_{\restriction X \dot{\to} Y}$ is one-to-one.

(95) For every function f such that f is one-to-one holds $(f_{\restriction X \dot{\to} Y})^{-1} = (f^{-1})_{\restriction Y \dot{\to} X}$.

(96) For every function f holds $f_{\restriction X \dot{\to} Y} \restriction Z = f_{\restriction X \cap Z \dot{\to} Y}$.

(97) For every function f holds $Z \restriction f_{\restriction X \dot{\to} Y} = f_{\restriction X \dot{\to} Z \cap Y}$.

Let us consider X, Y and let f be a relation between X and Y. We say that f is total if and only if:

(Def. 4) $\operatorname{dom} f = X$.

Next we state several propositions:

(99)[10] For every partial function f from X to Y such that f is total and $Y = \emptyset$ holds $X = \emptyset$.

(112)[11] Every partial function from $\emptyset$ to Y is total.

(113) For every function f such that $f_{\restriction X \dot{\to} Y}$ is total holds $X \subseteq \operatorname{dom} f$.

(114) If $\emptyset_{\restriction X \dot{\to} Y}$ is total, then $X = \emptyset$.

(115) For every function f such that $X \subseteq \operatorname{dom} f$ and $\operatorname{rng} f \subseteq Y$ holds $f_{\restriction X \dot{\to} Y}$ is total.

(116) For every function f such that $f_{\restriction X \dot{\to} Y}$ is total holds $f^{\circ} X \subseteq Y$.

(117) For every function f such that $X \subseteq \operatorname{dom} f$ and $f^{\circ} X \subseteq Y$ holds $f_{\restriction X \dot{\to} Y}$ is total.

Let us consider X, Y. The functor $X \dot{\to} Y$ yielding a set is defined by:

(Def. 5) $x \in X \dot{\to} Y$ iff there exists a function f such that $x = f$ and $\operatorname{dom} f \subseteq X$ and $\operatorname{rng} f \subseteq Y$.

Let us consider X, Y. Observe that $X \dot{\to} Y$ is non empty.

We now state several propositions:

(119)[12] For every partial function f from X to Y holds $f \in X \dot{\to} Y$.

(120) For every set f such that $f \in X \dot{\to} Y$ holds f is a partial function from X to Y.

(121) Every element of $X \dot{\to} Y$ is a partial function from X to Y.

(122) $\emptyset \dot{\to} Y = \{\emptyset\}$.

(123) $X \dot{\to} \emptyset = \{\emptyset\}$.

(125)[13] If $Z \subseteq X$, then $Z \dot{\to} Y \subseteq X \dot{\to} Y$.

(126) $\emptyset \dot{\to} Y \subseteq X \dot{\to} Y$.

(127) If $Z \subseteq Y$, then $X \dot{\to} Z \subseteq X \dot{\to} Y$.

(128) If $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$, then $X_1 \dot{\to} Y_1 \subseteq X_2 \dot{\to} Y_2$.

Let f, g be functions. The predicate $f \approx g$ is defined as follows:

(Def. 6) For every x such that $x \in \operatorname{dom} f \cap \operatorname{dom} g$ holds $f(x) = g(x)$.

Let us notice that the predicate $f \approx g$ is reflexive and symmetric.

The following propositions are true:

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[10] The proposition (98) has been removed.

[11] The propositions (100)–(111) have been removed.

[12] The proposition (118) has been removed.

[13] The proposition (124) has been removed.

(130)[14] For all functions f, g holds $f \approx g$ iff there exists a function h such that $f \cup g = h$.

(131) For all functions f, g holds $f \approx g$ iff there exists a function h such that $f \subseteq h$ and $g \subseteq h$.

(132) For all functions f, g such that $\operatorname{dom} f \subseteq \operatorname{dom} g$ holds $f \approx g$ iff for every x such that $x \in \operatorname{dom} f$ holds $f(x) = g(x)$.

(135)[15] For all functions f, g such that $f \subseteq g$ holds $f \approx g$.

(136) For all functions f, g such that $\operatorname{dom} f = \operatorname{dom} g$ and $f \approx g$ holds $f = g$.

(138)[16] For all functions f, g such that $\operatorname{dom} f$ misses $\operatorname{dom} g$ holds $f \approx g$.

(139) For all functions f, g, h such that $f \subseteq h$ and $g \subseteq h$ holds $f \approx g$.

(140) For all partial functions f, g from X to Y and for every function h such that $f \approx h$ and $g \subseteq f$ holds $g \approx h$.

(141) For every function f holds $\emptyset \approx f$.

(142) For every function f holds $\emptyset_{\restriction X \dot{\to} Y} \approx f$.

(143) For all partial functions f, g from X to $\{y\}$ holds $f \approx g$.

(144) For every function f holds $f \restriction X \approx f$.

(145) For every function f holds $Y \restriction f \approx f$.

(146) For every function f holds $Y \restriction f \restriction X \approx f$.

(147) For every function f holds $f_{\restriction X \dot{\to} Y} \approx f$.

(148) For all partial functions f, g from X to Y such that f is total and g is total and $f \approx g$ holds $f = g$.

(158)[17] For all partial functions f, g, h from X to Y such that $f \approx h$ and $g \approx h$ and h is total holds $f \approx g$.

(162)[18] Let f, g be partial functions from X to Y. Suppose if $Y = \emptyset$, then $X = \emptyset$ and $f \approx g$. Then there exists a partial function h from X to Y such that h is total and $f \approx h$ and $g \approx h$.

Let us consider X, Y and let f be a partial function from X to Y. The functor $\operatorname{TotFuncs} f$ yielding a set is defined as follows:

(Def. 7) $x \in \operatorname{TotFuncs} f$ iff there exists a partial function g from X to Y such that $g = x$ and g is total and $f \approx g$.

Next we state several propositions:

(168)[19] Let f be a partial function from X to Y and g be a set. If $g \in \operatorname{TotFuncs} f$, then g is a partial function from X to Y.

(169) For all partial functions f, g from X to Y such that $g \in \operatorname{TotFuncs} f$ holds g is total.

(171)[20] For every partial function f from X to Y and for every function g such that $g \in \operatorname{TotFuncs} f$ holds $f \approx g$.

(172) For every partial function f from X to $\emptyset$ such that $X \neq \emptyset$ holds $\operatorname{TotFuncs} f = \emptyset$.

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[14] The proposition (129) has been removed.
[15] The propositions (133) and (134) have been removed.
[16] The proposition (137) has been removed.
[17] The propositions (149)–(157) have been removed.
[18] The propositions (159)–(161) have been removed.
[19] The propositions (163)–(167) have been removed.
[20] The proposition (170) has been removed.

(174)[21] For every partial function f from X to Y holds f is total iff TotFuncs $f = \{f\}$.

(175) For every partial function f from $\emptyset$ to Y holds TotFuncs $f = \{f\}$.

(176) For every partial function f from $\emptyset$ to Y holds TotFuncs $f = \{\emptyset\}$.

(185)[22] For all partial functions f, g from X to Y such that TotFuncs f meets TotFuncs g holds $f \approx g$.

(186) For all partial functions f, g from X to Y such that if $Y = \emptyset$, then $X = \emptyset$ and $f \approx g$ holds TotFuncs f meets TotFuncs g.

2. APPENDIX

Let us consider X. Observe that there exists a binary relation on X which is total, reflexive, symmetric, antisymmetric, and transitive.

Let us observe that every binary relation which is symmetric and transitive is also reflexive.

Let us consider X. Note that id_X is symmetric, antisymmetric, and transitive.

Let us consider X. Then id_X is a total binary relation on X.

REFERENCES

[1] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[2] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[4] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[5] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[6] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[7] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

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[21] The proposition (173) has been removed.
[22] The propositions (177)–(184) have been removed.