

Processes in Petri nets

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Summary. Sequential and concurrent compositions of processes in Petri nets are introduced. A process is formalized as a set of (possible), so called, firing sequences. In the definition of the sequential composition the standard concatenation is used

$$R_1 \text{ before } R_2 = \{p_1 \frown p_2 : p_1 \in R_1 \wedge p_2 \in R_2\}$$

The definition of the concurrent composition is more complicated

$$R_1 \text{ concur } R_2 = \{q_1 \cup q_2 : q_1 \text{ misses } q_2 \wedge \text{Seq } q_1 \in R_1 \wedge \text{Seq } q_2 \in R_2\}$$

For example,

$$\{\langle t_0 \rangle\} \text{ concur } \{\langle t_1, t_2 \rangle\} = \{\langle t_0, t_1, t_2 \rangle, \langle t_1, t_0, t_2 \rangle, \langle t_1, t_2, t_0 \rangle\}$$

The basic properties of the compositions are shown.

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The articles [13], [12], [18], [5], [17], [9], [1], [3], [6], [11], [15], [2], [14], [7], [16], [8], [10], and [4] provide the notation and terminology for this paper.

1. PRELIMINARIES

We use the following convention: i denotes a natural number and x, x_1, x_2, y_1, y_2 denote sets.

The following propositions are true:

- (1) If $i > 0$, then $\{\langle i, x \rangle\}$ is a finite subsequence.
- (2) For every finite subsequence q holds $q = \emptyset$ iff $\text{Seq } q = \emptyset$.
- (3) For every finite subsequence q such that $q = \{\langle i, x \rangle\}$ holds $\text{Seq } q = \langle x \rangle$.

Let us observe that every finite subsequence is finite.

We now state several propositions:

- (4) For every finite subsequence q such that $\text{Seq } q = \langle x \rangle$ there exists i such that $q = \{\langle i, x \rangle\}$.
- (5) If $\{\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle\}$ is a finite sequence, then $x_1 = 1$ and $x_2 = 1$ and $y_1 = y_2$ or $x_1 = 1$ and $x_2 = 2$ or $x_1 = 2$ and $x_2 = 1$.

- (6) $\langle x_1, x_2 \rangle = \{\langle 1, x_1 \rangle, \langle 2, x_2 \rangle\}$.
- (7) For every finite subsequence p holds $\overline{\overline{p}} = \text{len Seq } p$.
- (8) For all binary relations P, R such that $\text{dom } P$ misses $\text{dom } R$ holds P misses R .
- (9) For all sets X, Y and for all binary relations P, R such that X misses Y holds $P \upharpoonright X$ misses $R \upharpoonright Y$.
- (10) For all functions f, g, h such that $f \subseteq h$ and $g \subseteq h$ and f misses g holds $\text{dom } f$ misses $\text{dom } g$.
- (11) For every set Y and for every binary relation R holds $Y \upharpoonright R \subseteq R \upharpoonright R^{-1}(Y)$.
- (12) For every set Y and for every function f holds $Y \upharpoonright f = f \upharpoonright f^{-1}(Y)$.

2. MARKINGS ON PETRI NETS

Let P be a set. A function is called a marking of P if:

(Def. 1) $\text{dom it} = P$ and $\text{rng it} \subseteq \mathbb{N}$.

We adopt the following rules: P, p, x denote sets, m_1, m_2, m_3, m_4, m denote markings of P , and i, j, j_1, k denote natural numbers.

Let P be a set, let m be a marking of P , and let p be a set. Then $m(p)$ is a natural number. We introduce the m multitude of p as a synonym of $m(p)$.

The scheme *MarkingLambda* deals with a set \mathcal{A} and a unary functor \mathcal{F} yielding a natural number, and states that:

There exists a marking m of \mathcal{A} such that for every p such that $p \in \mathcal{A}$ holds the m multitude of $p = \mathcal{F}(p)$

for all values of the parameters.

Let us consider P, m_1, m_2 . Let us observe that $m_1 = m_2$ if and only if:

(Def. 2) For every p such that $p \in P$ holds the m_1 multitude of $p =$ the m_2 multitude of p .

Let us consider P . The functor $\{\}_P$ yields a marking of P and is defined as follows:

(Def. 3) $\{\}_P = P \mapsto 0$.

Let P be a set and let m_1, m_2 be markings of P . The predicate $m_1 \subseteq m_2$ is defined as follows:

(Def. 4) For every set p such that $p \in P$ holds the m_1 multitude of $p \leq$ the m_2 multitude of p .

Let us note that the predicate $m_1 \subseteq m_2$ is reflexive.

The following propositions are true:

(13) $\{\}_P \subseteq m$.

(14) If $m_1 \subseteq m_2$ and $m_2 \subseteq m_3$, then $m_1 \subseteq m_3$.

Let P be a set and let m_1, m_2 be markings of P . The functor $m_1 + m_2$ yields a marking of P and is defined as follows:

(Def. 5) For every set p such that $p \in P$ holds the $m_1 + m_2$ multitude of $p =$ (the m_1 multitude of p) + (the m_2 multitude of p).

Let us note that the functor $m_1 + m_2$ is commutative.

Next we state the proposition

(15) $m + \{\}_P = m$.

Let P be a set and let m_1, m_2 be markings of P . Let us assume that $m_2 \subseteq m_1$. The functor $m_1 - m_2$ yields a marking of P and is defined as follows:

(Def. 6) For every set p such that $p \in P$ holds the $m_1 - m_2$ multitude of $p =$ (the m_1 multitude of p) $-$ (the m_2 multitude of p).

We now state a number of propositions:

- (16) $m_1 \subseteq m_1 + m_2$.
- (17) $m - \{ \}_P = m$.
- (18) If $m_1 \subseteq m_2$ and $m_2 \subseteq m_3$, then $m_3 - m_2 \subseteq m_3 - m_1$.
- (19) $(m_1 + m_2) - m_2 = m_1$.
- (20) If $m \subseteq m_1$ and $m_1 \subseteq m_2$, then $m_1 - m \subseteq m_2 - m$.
- (21) If $m_1 \subseteq m_2$, then $(m_2 + m_3) - m_1 = (m_2 - m_1) + m_3$.
- (22) If $m_1 \subseteq m_2$ and $m_2 \subseteq m_1$, then $m_1 = m_2$.
- (23) $(m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)$.
- (24) If $m_1 \subseteq m_2$ and $m_3 \subseteq m_4$, then $m_1 + m_3 \subseteq m_2 + m_4$.
- (25) If $m_1 \subseteq m_2$, then $m_2 - m_1 \subseteq m_2$.
- (26) If $m_1 \subseteq m_2$ and $m_3 \subseteq m_4$ and $m_4 \subseteq m_1$, then $m_1 - m_4 \subseteq m_2 - m_3$.
- (27) If $m_1 \subseteq m_2$, then $m_2 = (m_2 - m_1) + m_1$.
- (28) $(m_1 + m_2) - m_1 = m_2$.
- (29) If $m_2 + m_3 \subseteq m_1$, then $m_1 - m_2 - m_3 = m_1 - (m_2 + m_3)$.
- (30) If $m_3 \subseteq m_2$ and $m_2 \subseteq m_1$, then $m_1 - (m_2 - m_3) = (m_1 - m_2) + m_3$.
- (31) $m \in \mathbb{N}^P$.
- (32) If $x \in \mathbb{N}^P$, then x is a marking of P .

3. TRANSITIONS AND FIRING

Let us consider P . Transition of P is defined by:

(Def. 7) There exist m_1, m_2 such that it = $\langle m_1, m_2 \rangle$.

In the sequel t, t_1, t_2 denote transitions of P .

Let us consider P, t . Then t_1 is a marking of P . We introduce $\text{Pre } t$ as a synonym of t_1 . Then t_2 is a marking of P . We introduce $\text{Post } t$ as a synonym of t_2 .

Let us consider P, m, t . The functor $\text{fire}(t, m)$ yields a marking of P and is defined by:

(Def. 8) $\text{fire}(t, m) = \begin{cases} (m - \text{Pre } t) + \text{Post } t, & \text{if } \text{Pre } t \subseteq m, \\ m, & \text{otherwise.} \end{cases}$

The following proposition is true

- (33) If $\text{Pre } t_1 + \text{Pre } t_2 \subseteq m$, then $\text{fire}(t_2, \text{fire}(t_1, m)) = (m - \text{Pre } t_1 - \text{Pre } t_2) + \text{Post } t_1 + \text{Post } t_2$.

Let us consider P, t . The functor $\text{fire } t$ yielding a function is defined by:

(Def. 9) $\text{dom fire } t = \mathbb{N}^P$ and for every marking m of P holds $(\text{fire } t)(m) = \text{fire}(t, m)$.

We now state two propositions:

- (34) $\text{rng fire } t \subseteq \mathbb{N}^P$.

$$(35) \quad \text{fire}(t_2, \text{fire}(t_1, m)) = (\text{fire } t_2 \cdot \text{fire } t_1)(m).$$

Let us consider P . A non empty set is called a Petri net over P if:

(Def. 10) For every set x such that $x \in$ it holds x is a transition of P .

In the sequel N is a Petri net over P .

Let us consider P, N . We see that the element of N is a transition of P .

In the sequel e, e_1, e_2 are elements of N .

4. FIRING SEQUENCES OF TRANSITIONS

Let us consider P, N . A firing-sequence of N is an element of N^* .

In the sequel C, C_1, C_2 denote firing-sequences of N .

Let us consider P, N, C . The functor $\text{fire } C$ yields a function and is defined by the condition (Def. 11).

(Def. 11) There exists a function yielding finite sequence F such that $\text{fire } C = \text{compose}_{\mathbb{N}^P} F$ and $\text{len } F = \text{len } C$ and for every natural number i such that $i \in \text{dom } C$ holds $F(i) = \text{fire } (C_i \text{ qua element of } N)$.

The following propositions are true:

$$(36) \quad \text{fire } (\varepsilon_N) = \text{id}_{\mathbb{N}^P}.$$

$$(37) \quad \text{fire } \langle e \rangle = \text{fire } e.$$

$$(38) \quad \text{fire } e \cdot \text{id}_{\mathbb{N}^P} = \text{fire } e.$$

$$(39) \quad \text{fire } \langle e_1, e_2 \rangle = \text{fire } e_2 \cdot \text{fire } e_1.$$

$$(40) \quad \text{dom fire } C = \mathbb{N}^P \text{ and } \text{rng fire } C \subseteq \mathbb{N}^P.$$

$$(41) \quad \text{fire } (C_1 \wedge C_2) = \text{fire } C_2 \cdot \text{fire } C_1.$$

$$(42) \quad \text{fire } (C \wedge \langle e \rangle) = \text{fire } e \cdot \text{fire } C.$$

Let us consider P, N, C, m . The functor $\text{fire}(C, m)$ yielding a marking of P is defined by:

(Def. 12) $\text{fire}(C, m) = (\text{fire } C)(m)$.

5. SEQUENTIAL COMPOSITION

Let us consider P, N . A process in N is a subset of N^* .

In the sequel $R, R_1, R_2, R_3, P_1, P_2$ denote processes in N .

One can check that every function which is finite sequence-like is also finite subsequence-like.

Let us consider P, N, R_1, R_2 . The functor R_1 before R_2 yielding a process in N is defined by:

(Def. 13) $R_1 \text{ before } R_2 = \{C_1 \wedge C_2 : C_1 \in R_1 \wedge C_2 \in R_2\}$.

Let us consider P, N and let R_1, R_2 be non empty processes in N . Note that $R_1 \text{ before } R_2$ is non empty.

We now state several propositions:

$$(43) \quad (R_1 \cup R_2) \text{ before } R = (R_1 \text{ before } R) \cup (R_2 \text{ before } R).$$

$$(44) \quad R \text{ before } (R_1 \cup R_2) = (R \text{ before } R_1) \cup (R \text{ before } R_2).$$

$$(45) \quad \{C_1\} \text{ before } \{C_2\} = \{C_1 \wedge C_2\}.$$

$$(46) \quad \{C_1, C_2\} \text{ before } \{C\} = \{C_1 \wedge C, C_2 \wedge C\}.$$

$$(47) \quad \{C\} \text{ before } \{C_1, C_2\} = \{C \wedge C_1, C \wedge C_2\}.$$

6. CONCURRENT COMPOSITION

Let us consider P, N, R_1, R_2 . The functor $R_1 \text{ concur } R_2$ yielding a process in N is defined as follows:

(Def. 14) $R_1 \text{ concur } R_2 = \{C : \bigvee_{q_1, q_2: \text{finite subsequence}} (C = q_1 \cup q_2 \wedge q_1 \text{ misses } q_2 \wedge \text{Seq } q_1 \in R_1 \wedge \text{Seq } q_2 \in R_2)\}$.

Let us note that the functor $R_1 \text{ concur } R_2$ is commutative.

One can prove the following four propositions:

- (48) $(R_1 \cup R_2) \text{ concur } R = (R_1 \text{ concur } R) \cup (R_2 \text{ concur } R)$.
- (49) $\{\langle e_1 \rangle\} \text{ concur } \{\langle e_2 \rangle\} = \{\langle e_1, e_2 \rangle, \langle e_2, e_1 \rangle\}$.
- (50) $\{\langle e_1 \rangle, \langle e_2 \rangle\} \text{ concur } \{\langle e \rangle\} = \{\langle e_1, e \rangle, \langle e_2, e \rangle, \langle e, e_1 \rangle, \langle e, e_2 \rangle\}$.
- (51) $(R_1 \text{ before } R_2) \text{ before } R_3 = R_1 \text{ before } (R_2 \text{ before } R_3)$.

Let p be a finite subsequence and let i be a natural number. The functor $\text{Shift}^i p$ yielding a finite subsequence is defined as follows:

(Def. 15) $\text{dom } \text{Shift}^i p = \{i + k; k \text{ ranges over natural numbers: } k \in \text{dom } p\}$ and for every natural number j such that $j \in \text{dom } p$ holds $(\text{Shift}^i p)(i + j) = p(j)$.

In the sequel q, q_1, q_2 are finite subsequences.

Next we state a number of propositions:

- (52) $\text{Shift}^0 q = q$.
- (53) $\text{Shift}^{i+j} q = \text{Shift}^i \text{Shift}^j q$.
- (54) For every finite sequence p such that $p \neq \emptyset$ holds $\text{dom } \text{Shift}^i p = \{j_1 : i + 1 \leq j_1 \wedge j_1 \leq i + \text{len } p\}$.
- (55) For every finite subsequence q holds $q = \emptyset$ iff $\text{Shift}^i q = \emptyset$.
- (56) Let q be a finite subsequence. Then there exists a finite subsequence s_1 such that $\text{dom } s_1 = \text{dom } q$ and $\text{rng } s_1 = \text{dom } \text{Shift}^i q$ and for every k such that $k \in \text{dom } q$ holds $s_1(k) = i + k$ and s_1 is one-to-one.
- (57) For every finite subsequence q holds $\overline{\overline{q}} = \overline{\overline{\text{Shift}^i q}}$.
- (58) For every finite sequence p holds $\text{dom } p = \text{dom } \text{Seq } \text{Shift}^i p$.
- (59) For every finite sequence p such that $k \in \text{dom } p$ holds $(\text{Sgm } \text{dom } \text{Shift}^i p)(k) = i + k$.
- (60) For every finite sequence p such that $k \in \text{dom } p$ holds $(\text{Seq } \text{Shift}^i p)(k) = p(k)$.
- (61) For every finite sequence p holds $\text{Seq } \text{Shift}^i p = p$.

In the sequel p_1, p_2 are finite sequences.

One can prove the following propositions:

- (62) $\text{dom}(p_1 \cup \text{Shift}^{\text{len } p_1} p_2) = \text{Seg}(\text{len } p_1 + \text{len } p_2)$.
- (63) For every finite sequence p_1 and for every finite subsequence p_2 such that $\text{len } p_1 \leq i$ holds $\text{dom } p_1$ misses $\text{dom } \text{Shift}^i p_2$.
- (64) For all finite sequences p_1, p_2 holds $p_1 \hat{\ } p_2 = p_1 \cup \text{Shift}^{\text{len } p_1} p_2$.
- (65) For every finite sequence p_1 and for every finite subsequence p_2 such that $i \geq \text{len } p_1$ holds p_1 misses $\text{Shift}^i p_2$.
- (66) $(R_1 \text{ concur } R_2) \text{ concur } R_3 = R_1 \text{ concur } (R_2 \text{ concur } R_3)$.

- (67) R_1 before $R_2 \subseteq R_1$ concur R_2 .
- (68) If $R_1 \subseteq P_1$ and $R_2 \subseteq P_2$, then R_1 before $R_2 \subseteq P_1$ before P_2 .
- (69) If $R_1 \subseteq P_1$ and $R_2 \subseteq P_2$, then R_1 concur $R_2 \subseteq P_1$ concur P_2 .
- (70) For all finite subsequences p, q such that $q \subseteq p$ holds $\text{Shift}^i q \subseteq \text{Shift}^i p$.
- (71) For all finite sequences p_1, p_2 holds $\text{Shift}^{\text{len } p_1} p_2 \subseteq p_1 \hat{\ } p_2$.
- (72) If $\text{dom } q_1$ misses $\text{dom } q_2$, then $\text{dom } \text{Shift}^i q_1$ misses $\text{dom } \text{Shift}^i q_2$.
- (73) For all finite subsequences q, q_1, q_2 such that $q = q_1 \cup q_2$ and q_1 misses q_2 holds $\text{Shift}^i q_1 \cup \text{Shift}^i q_2 = \text{Shift}^i q$.
- (74) For every finite subsequence q holds $\text{dom } \text{Seq } q = \text{dom } \text{Seq } \text{Shift}^i q$.
- (75) For every finite subsequence q such that $k \in \text{dom } \text{Seq } q$ there exists j such that $j = (\text{Sgm } \text{dom } q)(k)$ and $(\text{Sgm } \text{dom } \text{Shift}^i q)(k) = i + j$.
- (76) For every finite subsequence q such that $k \in \text{dom } \text{Seq } q$ holds $(\text{Seq } \text{Shift}^i q)(k) = (\text{Seq } q)(k)$.
- (77) For every finite subsequence q holds $\text{Seq } q = \text{Seq } \text{Shift}^i q$.
- (78) For every finite subsequence q such that $\text{dom } q \subseteq \text{Seg } k$ holds $\text{dom } \text{Shift}^i q \subseteq \text{Seg}(i + k)$.
- (79) Let p be a finite sequence and q_1, q_2 be finite subsequences. If $q_1 \subseteq p$, then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \text{Shift}^{\text{len } p} q_2$.
- (80) Let p_1, p_2 be finite sequences and q_1, q_2 be finite subsequences. Suppose $q_1 \subseteq p_1$ and $q_2 \subseteq p_2$. Then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \text{Shift}^{\text{len } p_1} q_2$ and $\text{dom } \text{Seq } s_1 = \text{Seg}(\text{len } \text{Seq } q_1 + \text{len } \text{Seq } q_2)$.
- (81) Let p_1, p_2 be finite sequences and q_1, q_2 be finite subsequences. Suppose $q_1 \subseteq p_1$ and $q_2 \subseteq p_2$. Then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \text{Shift}^{\text{len } p_1} q_2$ and $\text{dom } \text{Seq } s_1 = \text{Seg}(\text{len } \text{Seq } q_1 + \text{len } \text{Seq } q_2)$ and $\text{Seq } s_1 = \text{Seq } q_1 \cup \text{Shift}^{\text{len } \text{Seq } q_1} \text{Seq } q_2$.
- (82) Let p_1, p_2 be finite sequences and q_1, q_2 be finite subsequences. Suppose $q_1 \subseteq p_1$ and $q_2 \subseteq p_2$. Then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \text{Shift}^{\text{len } p_1} q_2$ and $(\text{Seq } q_1) \hat{\ } \text{Seq } q_2 = \text{Seq } s_1$.
- (83) $(R_1$ concur $R_2)$ before $(P_1$ concur $P_2) \subseteq (R_1$ before $P_1)$ concur $(R_2$ before $P_2)$.

Let us consider P, N and let R_1, R_2 be non empty processes in N . Note that R_1 concur R_2 is non empty.

7. NEUTRAL PROCESS

Let us consider P and let N be a Petri net over P . The neutral process in N yields a non empty process in N and is defined as follows:

(Def. 16) The neutral process in $N = \{\epsilon_N\}$.

Let us consider P , let N be a Petri net over P , and let t be an element of N . The elementary process with t yielding a non empty process in N is defined by:

(Def. 17) The elementary process with $t = \{\langle t \rangle\}$.

The following propositions are true:

- (84) (The neutral process in N) before $R = R$.
- (85) R before the neutral process in $N = R$.
- (86) (The neutral process in N) concur $R = R$.

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