JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Integer and Rational Exponents

Konrad Raczkowski
Warsaw University
Białystok

Summary. The article includes definitios and theorems which are needed to define real exponent. The following notions are defined: natural exponent, integer exponent and rational exponent.

MML Identifier: PREPOWER.

WWW: http://mizar.org/JFM/Vol2/prepower.html

The articles [14], [2], [10], [4], [9], [1], [8], [3], [7], [6], [13], [12], [11], and [5] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: a, b, c are real numbers, m, n are natural numbers, k, l, i are integers, p, q are rational numbers, and s_1, s_2 are sequences of real numbers.

Next we state two propositions:

(2)[1] If s_1 is convergent and for every n holds $s_1(n) \geq a$, then $\lim s_1 \geq a$.

(3) If s_1 is convergent and for every n holds $s_1(n) \leq a$, then $\lim s_1 \leq a$.

Let a be a real number. The functor $(a^\kappa)_{\kappa \in \mathbb{N}}$ yields a sequence of real numbers and is defined as follows:

(Def. 1) For every m holds $(a^\kappa)_{\kappa \in \mathbb{N}}(m) = a^m$.

We now state two propositions:

(4) $s_1 = (a^\kappa)_{\kappa \in \mathbb{N}}$ iff $s_1(0) = 1$ and for every m holds $s_1(m+1) = s_1(m) \cdot a$.

(5) For every a such that $a \neq 0$ and for every m holds $(a^\kappa)_{\kappa \in \mathbb{N}}(m) \neq 0$.

Let a be a real number and let us consider n. Then a^n is a real number.

One can prove the following propositions:

(12)[2] If $0 \neq a$, then $0 \neq a^n$.

(13) If $0 < a$, then $0 < a^n$.

(14) $(\frac{1}{a})^n = \frac{1}{a^n}$.

(15) $(\frac{b}{a})^n = \frac{b^n}{a^n}$.

(17)[3] If $0 < a$ and $a \leq b$, then $a^n \leq b^n$.

[1] The proposition (1) has been removed.
[2] The propositions (6)–(11) have been removed.
[3] The proposition (16) has been removed.

(18) If $0 \leq a$ and $a < b$ and $1 \leq n$, then $a^n < b^n$.

(19) If $a \geq 1$, then $a^n \geq 1$.

(20) If $1 \leq a$ and $1 \leq n$, then $a \leq a^n$.

(21) If $1 < a$ and $2 \leq n$, then $a < a^n$.

(22) If $0 < a$ and $a \leq 1$ and $1 \leq n$, then $a^n \leq a$.

(23) If $0 < a$ and $a < 1$ and $2 \leq n$, then $a^n < a$.

(24) If $-1 < a$, then $(1+a)^n \geq 1 + n \cdot a$.

(25) If $0 < a$ and $a < 1$, then $(1+a)^n \leq 1 + 3^n \cdot a$.

(26) If s_1 is convergent and for every n holds $s_2(n) = s_1(n)^m$, then s_2 is convergent and $\lim s_2 = (\lim s_1)^m$.

Let us consider n and let a be a real number. Let us assume that $1 \leq n$. The functor $\mathrm{root}_n(a)$ yielding a real number is defined as follows:

(Def. 3)[4] (i) $(\mathrm{root}_n(a))^n = a$ and $\mathrm{root}_n(a) > 0$ if $a > 0$,

(ii) $\mathrm{root}_n(a) = 0$ if $a = 0$.

Let us consider n and let a be a real number. Then $\mathrm{root}_n(a)$ is a real number.

The following propositions are true:

(28)[5] If $a \geq 0$ and $n \geq 1$, then $(\mathrm{root}_n(a))^n = a$ and $\mathrm{root}_n(a^n) = a$.

(29) If $n \geq 1$, then $\mathrm{root}_n(1) = 1$.

(30) If $a \geq 0$, then $\mathrm{root}_1(a) = a$.

(31) If $a \geq 0$ and $b \geq 0$ and $n \geq 1$, then $\mathrm{root}_n(a \cdot b) = (\mathrm{root}_n(a)) \cdot (\mathrm{root}_n(b))$.

(32) If $a > 0$ and $n \geq 1$, then $\mathrm{root}_n(\frac{1}{a}) = \frac{1}{\mathrm{root}_n(a)}$.

(33) If $a \geq 0$ and $b > 0$ and $n \geq 1$, then $\mathrm{root}_n(\frac{a}{b}) = \frac{\mathrm{root}_n(a)}{\mathrm{root}_n(b)}$.

(34) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$, then $\mathrm{root}_n(\mathrm{root}_m(a)) = \mathrm{root}_{n \cdot m}(a)$.

(35) If $a \geq 0$ and $n \geq 1$ and $m \geq 1$, then $(\mathrm{root}_n(a)) \cdot (\mathrm{root}_m(a)) = \mathrm{root}_{n \cdot m}(a^{n+m})$.

(36) If $0 \leq a$ and $a \leq b$ and $n \geq 1$, then $\mathrm{root}_n(a) \leq \mathrm{root}_n(b)$.

(37) If $a \geq 0$ and $a < b$ and $n \geq 1$, then $\mathrm{root}_n(a) < \mathrm{root}_n(b)$.

(38) If $a \geq 1$ and $n \geq 1$, then $\mathrm{root}_n(a) \geq 1$ and $a \geq \mathrm{root}_n(a)$.

(39) If $0 \leq a$ and $a < 1$ and $n \geq 1$, then $a \leq \mathrm{root}_n(a)$ and $\mathrm{root}_n(a) < 1$.

(40) If $a > 0$ and $n \geq 1$, then $(\mathrm{root}_n(a)) - 1 \leq \frac{a-1}{n}$.

(41) If $a \geq 0$, then $\mathrm{root}_2(a) = \sqrt{a}$.

(42) Let s be a sequence of real numbers and given a. Suppose $a > 0$ and for every n such that $n \geq 1$ holds $s(n) = \mathrm{root}_n(a)$. Then s is convergent and $\lim s = 1$.

Let a be a real number and let us consider k. The functor $a^k_{\mathbb{Z}}$ is defined by:

[4] The definition (Def. 2) has been removed.

[5] The proposition (27) has been removed.

(Def. 4)(i) $a_{\mathbb{Z}}^{k} = a^{|k|}$ if $k \geq 0$,

(ii) $a_{\mathbb{Z}}^{k} = (a^{|k|})^{-1}$ if $k < 0$.

Let a be a real number and let us consider k. Note that $a_{\mathbb{Z}}^{k}$ is real.
Let a be a real number and let us consider k. Then $a_{\mathbb{Z}}^{k}$ is a real number.
We now state a number of propositions:

(44)[6] $a_{\mathbb{Z}}^{0} = 1$.

(45) $a_{\mathbb{Z}}^{1} = a$.

(46) If $a \neq 0$ and $i = n$, then $a_{\mathbb{Z}}^{i} = a^{n}$.

(47) $1_{\mathbb{Z}}^{k} = 1$.

(48) If $a \neq 0$, then $a_{\mathbb{Z}}^{k} \neq 0$.

(49) If $a > 0$, then $a_{\mathbb{Z}}^{k} > 0$.

(50) $(a \cdot b)_{\mathbb{Z}}^{k} = (a_{\mathbb{Z}}^{k}) \cdot b_{\mathbb{Z}}^{k}$.

(51) If $a \neq 0$, then $a_{\mathbb{Z}}^{-k} = \frac{1}{a_{\mathbb{Z}}^{k}}$.

(52) $(\frac{1}{a})_{\mathbb{Z}}^{k} = \frac{1}{a_{\mathbb{Z}}^{k}}$.

(53) If $a \neq 0$, then $a_{\mathbb{Z}}^{m-n} = \frac{a^{m}}{a^{n}}$.

(54) If $a \neq 0$, then $a_{\mathbb{Z}}^{k+l} = (a_{\mathbb{Z}}^{k}) \cdot a_{\mathbb{Z}}^{l}$.

(55) $(a_{\mathbb{Z}}^{k})_{\mathbb{Z}}^{l} = a_{\mathbb{Z}}^{k \cdot l}$.

(56) If $a > 0$ and $n \geq 1$, then $(\mathrm{root}_{n}(a))_{\mathbb{Z}}^{k} = \mathrm{root}_{n}(a_{\mathbb{Z}}^{k})$.

Let a be a real number and let us consider p. The functor $a_{\mathbb{Q}}^{p}$ is defined by:

(Def. 5) $a_{\mathbb{Q}}^{p} = \mathrm{root}_{\mathrm{den}\, p}(a_{\mathbb{Z}}^{\mathrm{num}\, p})$.

Let a be a real number and let us consider p. Observe that $a_{\mathbb{Q}}^{p}$ is real.
Let a be a real number and let us consider p. Then $a_{\mathbb{Q}}^{p}$ is a real number.
The following propositions are true:

(58)[7] If $a > 0$ and $p = 0$, then $a_{\mathbb{Q}}^{p} = 1$.

(59) If $a > 0$ and $p = 1$, then $a_{\mathbb{Q}}^{p} = a$.

(60) If $a > 0$ and $p = n$, then $a_{\mathbb{Q}}^{p} = a^{n}$.

(61) If $a > 0$ and $n \geq 1$ and $p = n^{-1}$, then $a_{\mathbb{Q}}^{p} = \mathrm{root}_{n}(a)$.

(62) $1_{\mathbb{Q}}^{p} = 1$.

(63) If $a > 0$, then $a_{\mathbb{Q}}^{p} > 0$.

(64) If $a > 0$, then $(a_{\mathbb{Q}}^{p}) \cdot a_{\mathbb{Q}}^{q} = a_{\mathbb{Q}}^{p+q}$.

(65) If $a > 0$, then $\frac{1}{a_{\mathbb{Q}}^{p}} = a_{\mathbb{Q}}^{-p}$.

(66) If $a > 0$, then $\frac{a_{\mathbb{Q}}^{p}}{a_{\mathbb{Q}}^{q}} = a_{\mathbb{Q}}^{p-q}$.

[6] The proposition (43) has been removed.
[7] The proposition (57) has been removed.

(67) If $a > 0$ and $b > 0$, then $(a \cdot b)^p_{\mathbb{Q}} = (a^p_{\mathbb{Q}}) \cdot b^p_{\mathbb{Q}}$.

(68) If $a > 0$, then $(\frac{1}{a})^p_{\mathbb{Q}} = \frac{1}{a^p_{\mathbb{Q}}}$.

(69) If $a > 0$ and $b > 0$, then $(\frac{a}{b})^p_{\mathbb{Q}} = \frac{a^p_{\mathbb{Q}}}{b^p_{\mathbb{Q}}}$.

(70) If $a > 0$, then $(a^p_{\mathbb{Q}})^q_{\mathbb{Q}} = a^{p \cdot q}_{\mathbb{Q}}$.

(71) If $a \geq 1$ and $p \geq 0$, then $a^p_{\mathbb{Q}} \geq 1$.

(72) If $a \geq 1$ and $p \leq 0$, then $a^p_{\mathbb{Q}} \leq 1$.

(73) If $a > 1$ and $p > 0$, then $a^p_{\mathbb{Q}} > 1$.

(74) If $a \geq 1$ and $p \geq q$, then $a^p_{\mathbb{Q}} \geq a^q_{\mathbb{Q}}$.

(75) If $a > 1$ and $p > q$, then $a^p_{\mathbb{Q}} > a^q_{\mathbb{Q}}$.

(76) If $a > 0$ and $a < 1$ and $p > 0$, then $a^p_{\mathbb{Q}} < 1$.

(77) If $a > 0$ and $a \leq 1$ and $p \leq 0$, then $a^p_{\mathbb{Q}} \geq 1$.

Let I_1 be a sequence of real numbers. We say that I_1 is rational sequence-like if and only if:

(Def. 6) For every n holds $I_1(n)$ is a rational number.

Let us observe that there exists a sequence of real numbers which is rational sequence-like.
A rational sequence is a rational sequence-like sequence of real numbers.
Let s be a rational sequence and let us consider n. Then $s(n)$ is a rational number.
One can prove the following propositions:

(79)[8] For every real number a there exists a rational sequence s such that s is convergent and $\lim s = a$ and for every n holds $s(n) \leq a$.

(80) For every a there exists a rational sequence s such that s is convergent and $\lim s = a$ and for every n holds $s(n) \geq a$.

Let a be a real number and let s be a rational sequence. The functor $a^s_{\mathbb{Q}}$ yielding a sequence of real numbers is defined by:

(Def. 7) For every n holds $(a^s_{\mathbb{Q}})(n) = a^{s(n)}_{\mathbb{Q}}$.

Next we state two propositions:

(82)[9] For every rational sequence s such that s is convergent and $a > 0$ holds $a^s_{\mathbb{Q}}$ is convergent.

(83) Let s_1, s_2 be rational sequences and given a. Suppose s_1 is convergent and s_2 is convergent and $\lim s_1 = \lim s_2$ and $a > 0$. Then $a^{s_1}_{\mathbb{Q}}$ is convergent and $a^{s_2}_{\mathbb{Q}}$ is convergent and $\lim(a^{s_1}_{\mathbb{Q}}) = \lim(a^{s_2}_{\mathbb{Q}})$.

Let a, b be real numbers. Let us assume that $a > 0$. The functor $a^b_{\mathbb{R}}$ yields a real number and is defined as follows:

(Def. 8) There exists a rational sequence s such that s is convergent and $\lim s = b$ and $a^s_{\mathbb{Q}}$ is convergent and $\lim(a^s_{\mathbb{Q}}) = a^b_{\mathbb{R}}$.

Let a, b be real numbers. Then $a^b_{\mathbb{R}}$ is a real number.
Next we state a number of propositions:

[8] The proposition (78) has been removed.
[9] The proposition (81) has been removed.

(85)[10] If $a > 0$, then $a_{\mathbb{R}}^{0} = 1$.

(86) If $a > 0$, then $a_{\mathbb{R}}^{1} = a$.

(87) $1_{\mathbb{R}}^{a} = 1$.

(88) If $a > 0$, then $a_{\mathbb{R}}^{p} = a_{\mathbb{Q}}^{p}$.

(89) If $a > 0$, then $a_{\mathbb{R}}^{b+c} = (a_{\mathbb{R}}^{b}) \cdot a_{\mathbb{R}}^{c}$.

(90) If $a > 0$, then $a_{\mathbb{R}}^{-c} = \frac{1}{a_{\mathbb{R}}^{c}}$.

(91) If $a > 0$, then $a_{\mathbb{R}}^{b-c} = \frac{a_{\mathbb{R}}^{b}}{a_{\mathbb{R}}^{c}}$.

(92) If $a > 0$ and $b > 0$, then $(a \cdot b)_{\mathbb{R}}^{c} = (a_{\mathbb{R}}^{c}) \cdot b_{\mathbb{R}}^{c}$.

(93) If $a > 0$, then $(\frac{1}{a})_{\mathbb{R}}^{c} = \frac{1}{a_{\mathbb{R}}^{c}}$.

(94) If $a > 0$ and $b > 0$, then $(\frac{a}{b})_{\mathbb{R}}^{c} = \frac{a_{\mathbb{R}}^{c}}{b_{\mathbb{R}}^{c}}$.

(95) If $a > 0$, then $a_{\mathbb{R}}^{b} > 0$.

(96) If $a \geq 1$ and $c \geq b$, then $a_{\mathbb{R}}^{c} \geq a_{\mathbb{R}}^{b}$.

(97) If $a > 1$ and $c > b$, then $a_{\mathbb{R}}^{c} > a_{\mathbb{R}}^{b}$.

(98) If $a > 0$ and $a \leq 1$ and $c \geq b$, then $a_{\mathbb{R}}^{c} \leq a_{\mathbb{R}}^{b}$.

(99) If $a \geq 1$ and $b \geq 0$, then $a_{\mathbb{R}}^{b} \geq 1$.

(100) If $a > 1$ and $b > 0$, then $a_{\mathbb{R}}^{b} > 1$.

(101) If $a \geq 1$ and $b \leq 0$, then $a_{\mathbb{R}}^{b} \leq 1$.

(102) If $a > 1$ and $b < 0$, then $a_{\mathbb{R}}^{b} < 1$.

(103) If s_1 is convergent and s_2 is convergent and $\lim s_1 > 0$ and for every n holds $s_1(n) > 0$ and $s_2(n) = s_1(n)_{\mathbb{Q}}^{p}$, then $\lim s_2 = (\lim s_1)_{\mathbb{Q}}^{p}$.

(104) If $a > 0$ and s_1 is convergent and s_2 is convergent and for every n holds $s_2(n) = a_{\mathbb{R}}^{s_1(n)}$, then $\lim s_2 = a_{\mathbb{R}}^{\lim s_1}$.

(105) If $a > 0$, then $(a_{\mathbb{R}}^{b})_{\mathbb{R}}^{c} = a_{\mathbb{R}}^{b \cdot c}$.

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Received September 21, 1990

Published January 2, 2004