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# The Fundamental Logic Structure in Quantum Mechanics

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**Summary.** In this article we present the logical structure given by four axioms of Mackey [4] in the set of propositions of Quantum Mechanics. The equivalence relation (PropRel(Q)) in the set of propositions (Prop Q) for given Quantum Mechanics Q is considered. The main text for this article is [6] where the structure of quotient space and the properties of equivalence relations, classes and partitions are studied.

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The articles [8], [3], [12], [10], [13], [14], [15], [1], [2], [11], [7], [5], [9], and [6] provide the notation and terminology for this paper.

In this paper $X_1$, $x$ are sets, $X$ is a non empty set, and $A$ is an event of the Borel sets.

Let us consider $X$ and let $S$ be a $\sigma$-field of subsets of $X$. The functor probabilities $S$ yielding a set is defined by:

(Def. 1) $x \in$ probabilities $S$ iff $x$ is a probability on $S$.

Let us consider $X$ and let $S$ be a $\sigma$-field of subsets of $X$. One can check that probabilities $S$ is non empty.

We consider quantum mechanics structures as systems

$\langle$ observables, control states, a probability $\rangle$,

where the observables and the control states constitute non empty sets and the probability is a function from [: the observables, the control states :] into probabilities (the Borel sets).

In the sequel $Q$ denotes a quantum mechanics structure.

Let us consider $Q$. The functor Obs $Q$ yielding a set is defined by:

(Def. 2) Obs $Q =$ the observables of $Q$.

The functor Sts $Q$ yields a set and is defined as follows:

(Def. 3) Sts $Q =$ the control states of $Q$.

Let us consider $Q$. One can verify that Obs $Q$ is non empty and Sts $Q$ is non empty.

In the sequel $A_1$ is an element of Obs $Q$, $s$ is an element of Sts $Q$, and $E$ is an event of the Borel sets.

Let us consider $Q$, $A_1$, $s$. The functor $\mathrm{Meas}(A_1, s)$ yields a probability on the Borel sets and is defined by:

(Def. 4) $\mathrm{Meas}(A_1, s) =$ (the probability of $Q$)$(\langle A_1, s \rangle)$.

Let $I_1$ be a quantum mechanics structure. We say that $I_1$ is quantum mechanics-like if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) For all elements $A_1$, $A_2$ of $\mathrm{Obs}\, I_1$ such that for every element $s$ of $\mathrm{Sts}\, I_1$ holds $\mathrm{Meas}(A_1,s) = \mathrm{Meas}(A_2,s)$ holds $A_1 = A_2$,

(ii) for all elements $s_1$, $s_2$ of $\mathrm{Sts}\, I_1$ such that for every element $A$ of $\mathrm{Obs}\, I_1$ holds $\mathrm{Meas}(A,s_1) = \mathrm{Meas}(A,s_2)$ holds $s_1 = s_2$, and

(iii) for all elements $s_1$, $s_2$ of $\mathrm{Sts}\, I_1$ and for every real number $t$ such that $0 \leq t$ and $t \leq 1$ there exists an element $s$ of $\mathrm{Sts}\, I_1$ such that for every element $A$ of $\mathrm{Obs}\, I_1$ and for every $E$ holds $\mathrm{Meas}(A,s)(E) = t \cdot \mathrm{Meas}(A,s_1)(E) + (1-t) \cdot \mathrm{Meas}(A,s_2)(E)$.

Let us note that there exists a quantum mechanics structure which is strict and quantum mechanics-like.

A quantum mechanics is a quantum mechanics-like quantum mechanics structure.

In the sequel $Q$ denotes a quantum mechanics and $s$ denotes an element of $\mathrm{Sts}\, Q$.

Let $X$ be a set. We consider POI structures over $X$ as systems

$\langle$ an ordering, an involution $\rangle$,

where the ordering is a binary relation on $X$ and the involution is a function from $X$ into $X$.

In the sequel $x_1$ is an element of $X_1$ and $I_2$ is a function from $X_1$ into $X_1$.

Let us consider $X_1$, $I_2$. We say that $I_2$ is an involution in $X_1$ if and only if:

(Def. 6) $I_2(I_2(x_1)) = x_1$.

Let us consider $X_1$ and let $W$ be a POI structure over $X_1$. We say that $W$ is a quantum logic on $X_1$ if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists a binary relation $O_1$ on $X_1$ and there exists a function $I_2$ from $X_1$ into $X_1$ such that

(i) $W = \langle O_1, I_2 \rangle$,

(ii) $O_1$ partially orders $X_1$,

(iii) $I_2$ is an involution in $X_1$, and

(iv) for all elements $x$, $y$ of $X_1$ such that $\langle x, y \rangle \in O_1$ holds $\langle I_2(y), I_2(x) \rangle \in O_1$.

Let us consider $Q$. The functor $\mathrm{Prop}\, Q$ yields a set and is defined as follows:

(Def. 8) $\mathrm{Prop}\, Q = [:\mathrm{Obs}\, Q, \text{the Borel sets}:]$.

Let us consider $Q$. Observe that $\mathrm{Prop}\, Q$ is non empty.

In the sequel $p$, $q$, $r$, $p_1$, $q_1$ denote elements of $\mathrm{Prop}\, Q$.

Let us consider $Q$, $p$. Then $\mathbf{p_1}$ is an element of $\mathrm{Obs}\, Q$. Then $\mathbf{p_2}$ is an event of the Borel sets.

We now state two propositions:

(14)[1] $p = \langle \mathbf{p_1}, \mathbf{p_2} \rangle$.

(16)[2] For every $E$ such that $E = (\mathbf{p_2})^{\mathrm{c}}$ holds $\mathrm{Meas}(\mathbf{p_1},s)(\mathbf{p_2}) = 1 - \mathrm{Meas}(\mathbf{p_1},s)(E)$.

Let us consider $Q$, $p$. The functor $\neg p$ yielding an element of $\mathrm{Prop}\, Q$ is defined by:

(Def. 9) $\neg p = \langle \mathbf{p_1}, (\mathbf{p_2})^{\mathrm{c}} \rangle$.

Let us consider $Q$, $p$, $q$. The predicate $p \vdash q$ is defined by:

(Def. 10) For every $s$ holds $\mathrm{Meas}(\mathbf{p_1},s)(\mathbf{p_2}) \leq \mathrm{Meas}(\mathbf{q_1},s)(\mathbf{q_2})$.

Let us consider $Q$, $p$, $q$. The predicate $p \equiv q$ is defined by:

(Def. 11) $p \vdash q$ and $q \vdash p$.

[1] The propositions (1)–(13) have been removed.

[2] The proposition (15) has been removed.

One can prove the following propositions:

(20)[3] $p \equiv q$ iff for every $s$ holds $\mathrm{Meas}(p_{\mathbf{1}}, s)(p_{\mathbf{2}}) = \mathrm{Meas}(q_{\mathbf{1}}, s)(q_{\mathbf{2}})$.

(21) $p \vdash p$.

(22) If $p \vdash q$ and $q \vdash r$, then $p \vdash r$.

(23) $p \equiv p$.

(24) If $p \equiv q$, then $q \equiv p$.

(25) If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

(26) $(\neg p)_{\mathbf{1}} = p_{\mathbf{1}}$ and $(\neg p)_{\mathbf{2}} = (p_{\mathbf{2}})^{\mathrm{c}}$.

(27) $\neg\neg p = p$.

(28) If $p \vdash q$, then $\neg q \vdash \neg p$.

Let us consider $Q$. The functor $\mathrm{PropRel}\, Q$ yielding an equivalence relation of $\mathrm{Prop}\, Q$ is defined by:

(Def. 12) $\langle p, q \rangle \in \mathrm{PropRel}\, Q$ iff $p \equiv q$.

In the sequel $B$, $C$ are subsets of $\mathrm{Prop}\, Q$.
Next we state the proposition

(30)[4] Let given $B$, $C$. Suppose $B \in \mathrm{Classes}\, \mathrm{PropRel}\, Q$ and $C \in \mathrm{Classes}\, \mathrm{PropRel}\, Q$. Let $a$, $b$, $c$, $d$ be elements of $\mathrm{Prop}\, Q$. If $a \in B$ and $b \in B$ and $c \in C$ and $d \in C$ and $a \vdash c$, then $b \vdash d$.

Let us consider $Q$. The functor $\mathrm{OrdRel}\, Q$ yields a binary relation on $\mathrm{Classes}\, \mathrm{PropRel}\, Q$ and is defined by:

(Def. 13) $\langle B, C \rangle \in \mathrm{OrdRel}\, Q$ iff $B \in \mathrm{Classes}\, \mathrm{PropRel}\, Q$ and $C \in \mathrm{Classes}\, \mathrm{PropRel}\, Q$ and for all $p$, $q$ such that $p \in B$ and $q \in C$ holds $p \vdash q$.

The following propositions are true:

(32)[5] $p \vdash q$ iff $\langle [p]_{\mathrm{PropRel}\, Q}, [q]_{\mathrm{PropRel}\, Q} \rangle \in \mathrm{OrdRel}\, Q$.

(33) For all $B$, $C$ such that $B \in \mathrm{Classes}\, \mathrm{PropRel}\, Q$ and $C \in \mathrm{Classes}\, \mathrm{PropRel}\, Q$ and for all $p_1$, $q_1$ such that $p_1 \in B$ and $q_1 \in B$ and $\neg p_1 \in C$ holds $\neg q_1 \in C$.

(34) For all $B$, $C$ such that $B \in \mathrm{Classes}\, \mathrm{PropRel}\, Q$ and $C \in \mathrm{Classes}\, \mathrm{PropRel}\, Q$ and for all $p$, $q$ such that $\neg p \in C$ and $\neg q \in C$ and $p \in B$ holds $q \in B$.

Let us consider $Q$. The functor $\mathrm{InvRel}\, Q$ yielding a function from $\mathrm{Classes}\, \mathrm{PropRel}\, Q$ into $\mathrm{Classes}\, \mathrm{PropRel}\, Q$ is defined by:

(Def. 14) $(\mathrm{InvRel}\, Q)([p]_{\mathrm{PropRel}\, Q}) = [\neg p]_{\mathrm{PropRel}\, Q}$.

We now state the proposition

(36)[6] For every $Q$ holds $\langle \mathrm{OrdRel}\, Q, \mathrm{InvRel}\, Q \rangle$ is a quantum logic on $\mathrm{Classes}\, \mathrm{PropRel}\, Q$.

[3] The propositions (17)–(19) have been removed.
[4] The proposition (29) has been removed.
[5] The proposition (31) has been removed.
[6] The proposition (35) has been removed.

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