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Introduction to Theory of Rearrangement[1]

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Summary. An introduction to the rearrangement theory for finite functions (e.g. with the finite domain and codomain). The notion of generators and cogenerators of finite sets (equivalent to the order in the language of finite sequences) has been defined. The notion of rearrangement for a function into finite set is presented. Some basic properties of these notions have been proved.

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The articles [16], [6], [19], [17], [20], [4], [3], [1], [9], [11], [2], [18], [21], [5], [12], [13], [7], [8], [10], [14], and [15] provide the notation and terminology for this paper.

In this paper n, m denote natural numbers and r denotes a real number.

Let D be a non empty set, let F be a partial function from D to $\mathbb{R}$, and let r be a real number. Then $r\,F$ is an element of $D \dot{\rightarrow} \mathbb{R}$.

Let I_1 be a finite sequence. We say that I_1 has cardinality by index if and only if:

(Def. 1) For every n such that $1 \leq n$ and $n \leq \operatorname{len} I_1$ and for every finite set B such that $B = I_1(n)$ holds $\operatorname{card} B = n$.

We say that I_1 is ascending if and only if:

(Def. 2) For every n such that $1 \leq n$ and $n \leq \operatorname{len} I_1 - 1$ holds $I_1(n) \subseteq I_1(n+1)$.

Let X be a set and let I_1 be a finite sequence of elements of X. We say that I_1 has length by cardinality if and only if:

(Def. 3) There exists a finite set B such that $B = \bigcup X$ and $\operatorname{len} I_1 = \operatorname{card} B$.

Let D be a non empty finite set. One can check that there exists a finite sequence of elements of 2^D which is ascending and has cardinality by index and length by cardinality.

Let D be a non empty finite set. A rearrangement generator of D is an ascending finite sequence of elements of 2^D with cardinality by index and length by cardinality.

In the sequel C, D denote non empty finite sets and a denotes a finite sequence of elements of 2^D.

We now state a number of propositions:

(1) For every finite sequence a of elements of 2^D holds a has length by cardinality iff $\operatorname{len} a = \operatorname{card} D$.

(2) Let a be a finite sequence. Then a is ascending if and only if for all n, m such that $n \leq m$ and $n \in \operatorname{dom} a$ and $m \in \operatorname{dom} a$ holds $a(n) \subseteq a(m)$.

[1]Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday.

(3) For every finite sequence a of elements of 2^D with cardinality by index and length by cardinality holds $a(\operatorname{len} a) = D$.

(4) For every finite sequence a of elements of 2^D with length by cardinality holds $\operatorname{len} a \neq 0$.

(5) Let a be an ascending finite sequence of elements of 2^D with cardinality by index and given n, m. If $n \in \operatorname{dom} a$ and $m \in \operatorname{dom} a$ and $n \neq m$, then $a(n) \neq a(m)$.

(6) Let a be an ascending finite sequence of elements of 2^D with cardinality by index and given n. If $1 \leq n$ and $n \leq \operatorname{len} a - 1$, then $a(n) \neq a(n+1)$.

(7) For every finite sequence a of elements of 2^D with cardinality by index such that $n \in \operatorname{dom} a$ holds $a(n) \neq \emptyset$.

(8) Let a be a finite sequence of elements of 2^D with cardinality by index. If $1 \leq n$ and $n \leq \operatorname{len} a - 1$, then $a(n+1) \setminus a(n) \neq \emptyset$.

(9) Let a be a finite sequence of elements of 2^D with cardinality by index and length by cardinality. Then there exists an element d of D such that $a(1) = \{d\}$.

(10) Let a be an ascending finite sequence of elements of 2^D with cardinality by index. Suppose $1 \leq n$ and $n \leq \operatorname{len} a - 1$. Then there exists an element d of D such that $a(n+1) \setminus a(n) = \{d\}$ and $a(n+1) = a(n) \cup \{d\}$ and $a(n+1) \setminus \{d\} = a(n)$.

Let D be a non empty finite set and let A be a rearrangement generator of D. The functor co-Gen(A) yields a rearrangement generator of D and is defined as follows:

(Def. 4) For every m such that $1 \leq m$ and $m \leq \operatorname{len}\text{co-Gen}(A) - 1$ holds $(\text{co-Gen}(A))(m) = D \setminus A(\operatorname{len} A - m)$.

Next we state two propositions:

(11) For every rearrangement generator A of D holds $\text{co-Gen}(\text{co-Gen}(A)) = A$.

(12) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{len}\text{MIM}(\text{FinS}(F,D)) = \operatorname{len}\text{CHI}(A,C)$.

Let D, C be non empty finite sets, let A be a rearrangement generator of C, and let F be a partial function from D to $\mathbb{R}$. The functor $F_A^{\wedge}$ yielding a partial function from C to $\mathbb{R}$ is defined as follows:

(Def. 5) $F_A^{\wedge} = \sum(\text{MIM}(\text{FinS}(F,D))\ \text{CHI}(A,C))$.

The functor $F_A^{\vee}$ yields a partial function from C to $\mathbb{R}$ and is defined as follows:

(Def. 6) $F_A^{\vee} = \sum(\text{MIM}(\text{FinS}(F,D))\ \text{CHI}(\text{co-Gen}(A),C))$.

One can prove the following propositions:

(13) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{dom} F_A^{\wedge} = C$.

(14) Let c be an element of C, F be a partial function from D to $\mathbb{R}$, and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then

(i) if $c \in A(1)$, then $(\text{MIM}(\text{FinS}(F,D))\ \text{CHI}(A,C))\#c = \text{MIM}(\text{FinS}(F,D))$, and

(ii) for every n such that $1 \leq n$ and $n < \operatorname{len} A$ and $c \in A(n+1) \setminus A(n)$ holds $(\text{MIM}(\text{FinS}(F,D))\ \text{CHI}(A,C))\#c = (n \mapsto (0\ \textbf{qua}\ \text{real number}))^\frown\text{MIM}((\text{FinS}(F,D))_{\restriction n})$.

(15) Let c be an element of C, F be a partial function from D to $\mathbb{R}$, and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then if $c \in A(1)$, then $(F_A^{\wedge})(c) = (\text{FinS}(F,D))(1)$ and for every n such that $1 \leq n$ and $n < \operatorname{len} A$ and $c \in A(n+1) \setminus A(n)$ holds $(F_A^{\wedge})(c) = (\text{FinS}(F,D))(n+1)$.

(16) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{rng} F_A^{\wedge} = \operatorname{rng} \operatorname{FinS}(F, D)$.

(17) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^{\wedge}$ and $\operatorname{FinS}(F, D)$ are fiberwise equipotent.

(18) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}(F_A^{\wedge}, C) = \operatorname{FinS}(F, D)$.

(19) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\sum_{\kappa=0}^{C} F_A^{\wedge}(\kappa) = \sum_{\kappa=0}^{D} F(\kappa)$.

(20) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}((F_A^{\wedge}) - r, C) = \operatorname{FinS}(F - r, D)$ and $\sum_{\kappa=0}^{C}((F_A^{\wedge}) - r)(\kappa) = \sum_{\kappa=0}^{D}(F - r)(\kappa)$.

(21) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{dom} F_A^{\vee} = C$.

(22) Let c be an element of C, F be a partial function from D to $\mathbb{R}$, and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then if $c \in (\text{co-Gen}(A))(1)$, then $(F_A^{\vee})(c) = (\operatorname{FinS}(F, D))(1)$ and for every n such that $1 \leq n$ and $n < \operatorname{len} \text{co-Gen}(A)$ and $c \in (\text{co-Gen}(A))(n+1) \setminus (\text{co-Gen}(A))(n)$ holds $(F_A^{\vee})(c) = (\operatorname{FinS}(F, D))(n+1)$.

(23) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{rng} F_A^{\vee} = \operatorname{rng} \operatorname{FinS}(F, D)$.

(24) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^{\vee}$ and $\operatorname{FinS}(F, D)$ are fiberwise equipotent.

(25) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}(F_A^{\vee}, C) = \operatorname{FinS}(F, D)$.

(26) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\sum_{\kappa=0}^{C} F_A^{\vee}(\kappa) = \sum_{\kappa=0}^{D} F(\kappa)$.

(27) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{FinS}((F_A^{\vee}) - r, C) = \operatorname{FinS}(F - r, D)$ and $\sum_{\kappa=0}^{C}((F_A^{\vee}) - r)(\kappa) = \sum_{\kappa=0}^{D}(F - r)(\kappa)$.

(28) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^{\vee}$ and $F_A^{\wedge}$ are fiberwise equipotent and $\operatorname{FinS}(F_A^{\vee}, C) = \operatorname{FinS}(F_A^{\wedge}, C)$ and $\sum_{\kappa=0}^{C} F_A^{\vee}(\kappa) = \sum_{\kappa=0}^{C} F_A^{\wedge}(\kappa)$.

(29) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_{+}((F_A^{\wedge}) - r)$ and $\max_{+}(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_{+}((F_A^{\wedge}) - r), C) = \operatorname{FinS}(\max_{+}(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_{+}((F_A^{\wedge}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_{+}(F - r)(\kappa)$.

(30) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_{-}((F_A^{\wedge}) - r)$ and $\max_{-}(F - r)$ are fiberwise equipotent and $\operatorname{FinS}(\max_{-}((F_A^{\wedge}) - r), C) = \operatorname{FinS}(\max_{-}(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_{-}((F_A^{\wedge}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_{-}(F - r)(\kappa)$.

(31) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$, then $\operatorname{len} \operatorname{FinS}(F_A^{\wedge}, C) = \operatorname{card} C$ and $1 \leq \operatorname{len} \operatorname{FinS}(F_A^{\wedge}, C)$.

(32) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$ and $n \in \operatorname{dom} A$, then $\operatorname{FinS}(F_A^{\wedge}, C) \restriction n = \operatorname{FinS}(F_A^{\wedge}, A(n))$.

(33) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$, then $(F - r)_A^{\wedge} = (F_A^{\wedge}) - r$.

(34) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_+((F_A^{\vee}) - r)$ and $\max_+(F - r)$ are fiberwise equipotent and $\mathrm{FinS}(\max_+((F_A^{\vee}) - r), C) = \mathrm{FinS}(\max_+(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_+((F_A^{\vee}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_+(F - r)(\kappa)$.

(35) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_-((F_A^{\vee}) - r)$ and $\max_-(F - r)$ are fiberwise equipotent and $\mathrm{FinS}(\max_-((F_A^{\vee}) - r), C) = \mathrm{FinS}(\max_-(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_-((F_A^{\vee}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_-(F - r)(\kappa)$.

(36) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$, then $\operatorname{len} \mathrm{FinS}(F_A^{\vee}, C) = \operatorname{card} C$ and $1 \leq \operatorname{len} \mathrm{FinS}(F_A^{\vee}, C)$.

(37) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$ and $n \in \operatorname{dom} A$, then $\mathrm{FinS}(F_A^{\vee}, C) \restriction n = \mathrm{FinS}(F_A^{\vee}, (\text{co-Gen}(A))(n))$.

(38) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. If F is total and $\operatorname{card} D = \operatorname{card} C$, then $(F - r)_A^{\vee} = (F_A^{\vee}) - r$.

(39) Let F be a partial function from D to $\mathbb{R}$ and A be a rearrangement generator of C. Suppose F is total and $\operatorname{card} D = \operatorname{card} C$. Then $F_A^{\wedge}$ and F are fiberwise equipotent and $F_A^{\vee}$ and F are fiberwise equipotent and $\operatorname{rng} F_A^{\wedge} = \operatorname{rng} F$ and $\operatorname{rng} F_A^{\vee} = \operatorname{rng} F$.

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