JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Operations on Submodules in Right Module over Associative Ring

Michał Muzalewski
Warsaw University
Białystok

Wojciech Skaba
Nicolaus Copernicus University
Toruń

MML Identifier: RMOD_3.

WWW: http://mizar.org/JFM/Vol2/rmod_3.html

The articles [7], [3], [9], [1], [10], [2], [11], [8], [6], [4], and [5] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: R denotes a ring, V denotes a right module over R, W, W_1, W_2, W_3 denote submodules of V, u, u_1, u_2, v, v_1, v_2 denote vectors of V, and x denotes a set.

Let us consider R, let us consider V, and let us consider W_1, W_2. The functor $W_1 + W_2$ yields a strict submodule of V and is defined as follows:

(Def. 1) The carrier of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

Let us consider R, let us consider V, and let us consider W_1, W_2. The functor $W_1 \cap W_2$ yielding a strict submodule of V is defined as follows:

(Def. 2) The carrier of $W_1 \cap W_2 =$ (the carrier of $W_1) \cap ($the carrier of W_2).

We now state a number of propositions:

(5)[1] $x \in W_1 + W_2$ iff there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ iff $x \in W_1$ and $x \in W_2$.

(8) For every strict submodule W of V holds $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(11) W_1 is a submodule of $W_1 + W_2$ and W_2 is a submodule of $W_1 + W_2$.

(12) For every strict submodule W_2 of V holds W_1 is a submodule of W_2 iff $W_1 + W_2 = W_2$.

(13) For every strict submodule W of V holds $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

(14) For every strict right module V over R holds $\mathbf{0}_V + \Omega_V = V$ and $\Omega_V + \mathbf{0}_V = V$.

[1] The propositions (1)–(4) have been removed.

(15) Let V be a right module over R and W be a submodule of V. Then $\Omega_V + W =$ the right module structure of V and $W + \Omega_V =$ the right module structure of V.

(16) For every strict right module V over R holds $\Omega_V + \Omega_V = V$.

(17) For every strict submodule W of V holds $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(20) $W_1 \cap W_2$ is a submodule of W_1 and $W_1 \cap W_2$ is a submodule of W_2.

(21)(i) For every strict submodule W_1 of V such that W_1 is a submodule of W_2 holds $W_1 \cap W_2 = W_1$, and

(ii) for every W_1 such that $W_1 \cap W_2 = W_1$ holds W_1 is a submodule of W_2.

(22) If W_1 is a submodule of W_2, then $W_1 \cap W_3$ is a submodule of $W_2 \cap W_3$.

(23) If W_1 is a submodule of W_3, then $W_1 \cap W_2$ is a submodule of W_3.

(24) If W_1 is a submodule of W_2 and a submodule of W_3, then W_1 is a submodule of $W_2 \cap W_3$.

(25) $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(27)[2] For every strict submodule W of V holds $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(28) For every strict right module V over R holds $\Omega_V \cap \Omega_V = V$.

(29) $W_1 \cap W_2$ is a submodule of $W_1 + W_2$.

(30) For every strict submodule W_2 of V holds $W_1 \cap W_2 + W_2 = W_2$.

(31) For every strict submodule W_1 of V holds $W_1 \cap (W_1 + W_2) = W_1$.

(32) $W_1 \cap W_2 + W_2 \cap W_3$ is a submodule of $W_2 \cap (W_1 + W_3)$.

(33) If W_1 is a submodule of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(34) $W_2 + W_1 \cap W_3$ is a submodule of $(W_1 + W_2) \cap (W_2 + W_3)$.

(35) If W_1 is a submodule of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(36) For every strict submodule W_1 of V such that W_1 is a submodule of W_3 holds $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(37) For all strict submodules W_1, W_2 of V holds $W_1 + W_2 = W_2$ iff $W_1 \cap W_2 = W_1$.

(38) For all strict submodules W_2, W_3 of V such that W_1 is a submodule of W_2 holds $W_1 + W_3$ is a submodule of $W_2 + W_3$.

(39) If W_1 is a submodule of W_2, then W_1 is a submodule of $W_2 + W_3$.

(40) If W_1 is a submodule of W_3 and W_2 is a submodule of W_3, then $W_1 + W_2$ is a submodule of W_3.

(41) There exists W such that the carrier of $W =$ (the carrier of W_1) $\cup$ (the carrier of W_2) if and only if W_1 is a submodule of W_2 or W_2 is a submodule of W_1.

Let us consider R and let us consider V. The functor $\mathrm{Sub}(V)$ yielding a set is defined by:

(Def. 3) For every x holds $x \in \mathrm{Sub}(V)$ iff there exists a strict submodule W of V such that $W = x$.

[2] The proposition (26) has been removed.

Let us consider R and let us consider V. Observe that $\mathrm{Sub}(V)$ is non empty.
We now state the proposition

(44)[3] For every strict right module V over R holds $V \in \mathrm{Sub}(V)$.

Let us consider R, let us consider V, and let us consider W_1, W_2. We say that V is the direct sum of W_1 and W_2 if and only if:

(Def. 4) The right module structure of $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

The following two propositions are true:

(46)[4] If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(47) Every strict right module V over R is the direct sum of $\mathbf{0}_V$ and Ω_V and the direct sum of Ω_V and $\mathbf{0}_V$.

In the sequel C_1 is a coset of W_1 and C_2 is a coset of W_2.
The following propositions are true:

(48) If C_1 meets C_2, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(49) Let V be a right module over R and W_1, W_2 be submodules of V. Then V is the direct sum of W_1 and W_2 if and only if for every coset C_1 of W_1 and for every coset C_2 of W_2 there exists a vector v of V such that $C_1 \cap C_2 = \{v\}$.

(50) Let V be a strict right module over R and W_1, W_2 be submodules of V. Then $W_1 + W_2 = V$ if and only if for every vector v of V there exist vectors v_1, v_2 of V such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(51) Let V be a right module over R, W_1, W_2 be submodules of V, and v, v_1, v_2, u_1, u_2 be vectors of V. Suppose V is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$. Then $v_1 = u_1$ and $v_2 = u_2$.

(52) Suppose $V = W_1 + W_2$ and there exists v such that for all v_1, v_2, u_1, u_2 such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$. Then V is the direct sum of W_1 and W_2.

Let us consider R, let V be a right module over R, let v be a vector of V, and let W_1, W_2 be submodules of V. Let us assume that V is the direct sum of W_1 and W_2. The functor $v_{\langle W_1, W_2 \rangle}$ yielding an element of $[:$ the carrier of V, the carrier of $V :]$ is defined as follows:

(Def. 5) $v = (v_{\langle W_1,W_2\rangle})_{\mathbf{1}} + (v_{\langle W_1,W_2\rangle})_{\mathbf{2}}$ and $(v_{\langle W_1,W_2\rangle})_{\mathbf{1}} \in W_1$ and $(v_{\langle W_1,W_2\rangle})_{\mathbf{2}} \in W_2$.

Next we state two propositions:

(57)[5] If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2\rangle})_{\mathbf{1}} = (v_{\langle W_2,W_1\rangle})_{\mathbf{2}}$.

(58) If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2\rangle})_{\mathbf{2}} = (v_{\langle W_2,W_1\rangle})_{\mathbf{1}}$.

In the sequel A_1, A_2 are elements of $\mathrm{Sub}(V)$.
Let us consider R and let us consider V. The functor $\mathrm{SubJoin}V$ yields a binary operation on $\mathrm{Sub}(V)$ and is defined by:

(Def. 6) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}V)(A_1, A_2) = W_1 + W_2$.

[3] The propositions (42) and (43) have been removed.
[4] The proposition (45) has been removed.
[5] The propositions (53)–(56) have been removed.

Let us consider R and let us consider V. The functor SubMeet$V$ yielding a binary operation on $\mathrm{Sub}(V)$ is defined by:

(Def. 7) For all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}V)(A_1, A_2) = W_1 \cap W_2$.

We now state several propositions:

(63)[6] $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is a lattice.

(64) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is a lower bound lattice.

(65) For every right module V over R holds $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is an upper bound lattice.

(66) For every right module V over R holds $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is a bound lattice.

(67) $\langle \mathrm{Sub}(V), \mathrm{SubJoin}V, \mathrm{SubMeet}V \rangle$ is a modular lattice.

REFERENCES

[1] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[2] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[3] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[4] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/vectsp_2.html.

[5] Michał Muzalewski and Wojciech Skaba. Submodules and cosets of submodules in right module over associative ring. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rmod_2.html.

[6] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/domain_1.html.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[8] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[9] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[10] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[11] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/lattices.html.

Received October 22, 1990

Published January 2, 2004

[6] The propositions (59)–(62) have been removed.