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Remarks on Special Subsets of Topological Spaces

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Summary. Let X be a topological space and let A be a subset of X. Recall that A is *nowhere dense* in X if its closure is a boundary subset of X, i.e., if $\mathrm{Int}\overline{A} = \emptyset$ (see [2]). We introduce here the concept of everywhere dense subsets in X, which is dual to the above one. Namely, A is said to be *everywhere dense* in X if its interior is a dense subset of X, i.e., if $\overline{\mathrm{Int}A} =$ the carrier of X.

Our purpose is to list a number of properties of such sets (comp. [7]). As a sample we formulate their two dual characterizations. The first one characterizes thin sets in X : *A is nowhere dense iff for every open nonempty subset G of X there is an open nonempty subset of X contained in G and disjoint from A.* The corresponding second one characterizes thick sets in X : *A is everywhere dense iff for every closed subset F of X distinct from the carrier of X there is a closed subset of X distinct from the carrier of X, which contains F and together with A covers the carrier of X.* We also give some connections between both these concepts. Of course, *A is everywhere (nowhere) dense in X iff its complement is nowhere (everywhere) dense*. Moreover, *A is nowhere dense iff there are two subsets of X, C boundary closed and B everywhere dense, such that $A = C \cap B$ and $C \cup B$ covers the carrier of X.* Dually, *A is everywhere dense iff there are two disjoint subsets of X, C open dense and B nowhere dense, such that $A = C \cup B$.*

Note that some relationships between everywhere (nowhere) dense sets in X and everywhere (nowhere) dense sets in subspaces of X are also indicated.

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The articles [4], [6], [3], [7], [5], and [1] provide the notation and terminology for this paper.

1. Selected Properties of Subsets of a Topological Space

In this paper X denotes a topological structure and A denotes a subset of X.

We now state three propositions:

(1) $A = \emptyset_X$ iff $A^{\rm c} = \Omega_X$ and $A = \emptyset$ iff $A^{\rm c} =$ the carrier of X.

(2) $A = \Omega_X$ iff $A^{\rm c} = \emptyset_X$ and $A =$ the carrier of X iff $A^{\rm c} = \emptyset$.

(3) For every topological space X and for all subsets A, B of X holds $\mathrm{Int}A \cap \overline{B} \subseteq \overline{A \cap B}$.

In the sequel X denotes a topological space and A, B denote subsets of X.

We now state several propositions:

(4) $\mathrm{Int}(A \cup B) \subseteq \overline{A} \cup \mathrm{Int}B$.

(5) For every subset A of X such that A is closed holds $\mathrm{Int}(A \cup B) \subseteq A \cup \mathrm{Int}B$.

(6) For every subset A of X such that A is closed holds $\mathrm{Int}(A \cup B) = \mathrm{Int}(A \cup \mathrm{Int}\,B)$.

(7) If A misses $\mathrm{Int}\overline{A}$, then $\mathrm{Int}\overline{A} = \emptyset$.

(8) If $A \cup \overline{\mathrm{Int}A}$ = the carrier of X, then $\overline{\mathrm{Int}A}$ = the carrier of X.

2. Special Subsets of a Topological Space

Let X be a topological structure and let A be a subset of X. Let us observe that A is boundary if and only if:

(Def. 1) $\mathrm{Int}A = \emptyset$.

We now state the proposition

(9) $\emptyset_X$ is boundary.

In the sequel X is a non empty topological space and A is a subset of X.
Next we state the proposition

(10) If A is boundary, then $A \neq$ the carrier of X.

In the sequel X is a topological space and A, B are subsets of X.
We now state several propositions:

(11) If B is boundary and $A \subseteq B$, then A is boundary.

(12) A is boundary iff for every subset C of X such that $A^{\mathrm{c}} \subseteq C$ and C is closed holds C = the carrier of X.

(13) A is boundary iff for every subset G of X such that $G \neq \emptyset$ and G is open holds A^{c} meets G.

(14) A is boundary iff for every subset F of X such that F is closed holds $\mathrm{Int}F = \mathrm{Int}(F \cup A)$.

(15) If A is boundary or B is boundary, then $A \cap B$ is boundary.

Let X be a topological structure and let A be a subset of X. Let us observe that A is dense if and only if:

(Def. 2) $\overline{A}$ = the carrier of X.

Next we state the proposition

(16) Ω_X is dense.

In the sequel X denotes a non empty topological space and A, B denote subsets of X.
The following propositions are true:

(17) If A is dense, then $A \neq \emptyset$.

(18) A is dense iff A^{c} is boundary.

(19) A is dense iff for every subset C of X such that $A \subseteq C$ and C is closed holds C = the carrier of X.

(20) A is dense iff for every subset G of X such that G is open holds $\overline{G} = \overline{G \cap A}$.

(21) If A is dense or B is dense, then $A \cup B$ is dense.

Let X be a topological structure and let A be a subset of X. Let us observe that A is nowhere dense if and only if:

(Def. 3) $\mathrm{Int}\overline{A} = \emptyset$.

The following propositions are true:

(22) $\emptyset_X$ is nowhere dense.

(23) If A is nowhere dense, then $A \neq$ the carrier of X.

(24) If A is nowhere dense, then $\overline{A}$ is nowhere dense.

(25) If A is nowhere dense, then A is not dense.

(26) If B is nowhere dense and $A \subseteq B$, then A is nowhere dense.

(27) A is nowhere dense iff there exists a subset C of X such that $A \subseteq C$ and C is closed and boundary.

(28) A is nowhere dense if and only if for every subset G of X such that $G \neq \emptyset$ and G is open there exists a subset H of X such that $H \subseteq G$ and $H \neq \emptyset$ and H is open and A misses H.

(29) If A is nowhere dense or B is nowhere dense, then $A \cap B$ is nowhere dense.

(30) If A is nowhere dense and B is boundary, then $A \cup B$ is boundary.

Let X be a topological structure and let A be a subset of X. We say that A is everywhere dense if and only if:

(Def. 4) $\overline{\mathrm{Int} A} = \Omega_X$.

Let X be a topological structure and let A be a subset of X. Let us observe that A is everywhere dense if and only if:

(Def. 5) $\overline{\mathrm{Int} A} =$ the carrier of X.

We now state a number of propositions:

(31) Ω_X is everywhere dense.

(32) If A is everywhere dense, then $\mathrm{Int} A$ is everywhere dense.

(33) If A is everywhere dense, then A is dense.

(34) If A is everywhere dense, then $A \neq \emptyset$.

(35) A is everywhere dense iff $\mathrm{Int} A$ is dense.

(36) If A is open and dense, then A is everywhere dense.

(37) If A is everywhere dense, then A is not boundary.

(38) If A is everywhere dense and $A \subseteq B$, then B is everywhere dense.

(39) A is everywhere dense iff A^{c} is nowhere dense.

(40) A is nowhere dense iff A^{c} is everywhere dense.

(41) A is everywhere dense iff there exists a subset C of X such that $C \subseteq A$ and C is open and dense.

(42) A is everywhere dense if and only if for every subset F of X such that $F \neq$ the carrier of X and F is closed there exists a subset H of X such that $F \subseteq H$ and $H \neq$ the carrier of X and H is closed and $A \cup H =$ the carrier of X.

(43) If A is everywhere dense or B is everywhere dense, then $A \cup B$ is everywhere dense.

(44) If A is everywhere dense and B is everywhere dense, then $A \cap B$ is everywhere dense.

(45) If A is everywhere dense and B is dense, then $A \cap B$ is dense.

(46) If A is dense and B is nowhere dense, then $A \setminus B$ is dense.

(47) If A is everywhere dense and B is boundary, then $A \setminus B$ is dense.

(48) If A is everywhere dense and B is nowhere dense, then $A \setminus B$ is everywhere dense.

In the sequel D is a subset of X.
The following four propositions are true:

(49) Suppose D is everywhere dense. Then there exist subsets C, B of X such that C is open and dense and B is nowhere dense and $C \cup B = D$ and C misses B.

(50) Suppose D is everywhere dense. Then there exist subsets C, B of X such that C is open and dense and B is closed and boundary and $C \cup D \cap B = D$ and C misses B and $C \cup B =$ the carrier of X.

(51) Suppose D is nowhere dense. Then there exist subsets C, B of X such that C is closed and boundary and B is everywhere dense and $C \cap B = D$ and $C \cup B =$ the carrier of X.

(52) Suppose D is nowhere dense. Then there exist subsets C, B of X such that C is closed and boundary and B is open and dense and $C \cap (D \cup B) = D$ and C misses B and $C \cup B =$ the carrier of X.

3. Properties of Subsets in Subspaces

In the sequel Y_0 is a subspace of X.
The following propositions are true:

(53) For every subset A of X and for every subset B of Y_0 such that $B \subseteq A$ holds $\overline{B} \subseteq \overline{A}$.

(54) Let C, A be subsets of X and B be a subset of Y_0. If C is closed and $C \subseteq$ the carrier of Y_0 and $A \subseteq C$ and $A = B$, then $\overline{A} = \overline{B}$.

(55) Let Y_0 be a closed non empty subspace of X, A be a subset of X, and B be a subset of Y_0. If $A = B$, then $\overline{A} = \overline{B}$.

(56) For every subset A of X and for every subset B of Y_0 such that $A \subseteq B$ holds $\operatorname{Int} A \subseteq \operatorname{Int} B$.

(57) Let Y_0 be a non empty subspace of X, C, A be subsets of X, and B be a subset of Y_0. If C is open and $C \subseteq$ the carrier of Y_0 and $A \subseteq C$ and $A = B$, then $\operatorname{Int} A = \operatorname{Int} B$.

(58) Let Y_0 be an open non empty subspace of X, A be a subset of X, and B be a subset of Y_0. If $A = B$, then $\operatorname{Int} A = \operatorname{Int} B$.

In the sequel X_0 is a subspace of X.
Next we state two propositions:

(59) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if A is dense, then B is dense.

(60) Let C, A be subsets of X and B be a subset of X_0. Suppose $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$. Then C is dense and B is dense if and only if A is dense.

In the sequel X_0 denotes a non empty subspace of X.
Next we state a number of propositions:

(61) Let A be a subset of X and B be a subset of X_0. If $A \subseteq B$, then if A is everywhere dense, then B is everywhere dense.

(62) Let C, A be subsets of X and B be a subset of X_0. Suppose C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$. Then C is dense and B is everywhere dense if and only if A is everywhere dense.

(63) Let X_0 be an open non empty subspace of X, A, C be subsets of X, and B be a subset of X_0. Suppose $C =$ the carrier of X_0 and $A = B$. Then C is dense and B is everywhere dense if and only if A is everywhere dense.

(64) Let C, A be subsets of X and B be a subset of X_0. Suppose $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$. Then C is everywhere dense and B is everywhere dense if and only if A is everywhere dense.

(65) For every subset A of X and for every subset B of X_0 such that $A \subseteq B$ holds if B is boundary, then A is boundary.

(66) Let C, A be subsets of X and B be a subset of X_0. Suppose C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$. If A is boundary, then B is boundary.

(67) Let X_0 be an open non empty subspace of X, A be a subset of X, and B be a subset of X_0. If $A = B$, then A is boundary iff B is boundary.

(68) Let A be a subset of X and B be a subset of X_0. If $A \subseteq B$, then if B is nowhere dense, then A is nowhere dense.

(69) Let C, A be subsets of X and B be a subset of X_0. Suppose C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$. If A is nowhere dense, then B is nowhere dense.

(70) Let X_0 be an open non empty subspace of X, A be a subset of X, and B be a subset of X_0. If $A = B$, then A is nowhere dense iff B is nowhere dense.

4. Subsets in Topological Spaces with the same Topological Structures

We now state the proposition

(71) Let X_1, X_2 be 1-sorted structures. Suppose the carrier of $X_1 =$ the carrier of X_2. Let C_1 be a subset of X_1 and C_2 be a subset of X_2. Then $C_1 = C_2$ if and only if ${C_1}^{c} = {C_2}^{c}$.

In the sequel X_1, X_2 are topological structures.

Next we state two propositions:

(72) Suppose that

(i) the carrier of $X_1 =$ the carrier of X_2, and

(ii) for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds C_1 is open iff C_2 is open.

Then the topological structure of $X_1 =$ the topological structure of X_2.

(73) Suppose that

(i) the carrier of $X_1 =$ the carrier of X_2, and

(ii) for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds C_1 is closed iff C_2 is closed.

Then the topological structure of $X_1 =$ the topological structure of X_2.

In the sequel X_1, X_2 are topological spaces.

One can prove the following propositions:

(74) Suppose that

(i) the carrier of $X_1 =$ the carrier of X_2, and

(ii) for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds $\mathrm{Int} C_1 = \mathrm{Int} C_2$.

Then the topological structure of $X_1 =$ the topological structure of X_2.

(75) Suppose that

(i) the carrier of X_1 = the carrier of X_2, and

(ii) for every subset C_1 of X_1 and for every subset C_2 of X_2 such that $C_1 = C_2$ holds $\overline{C_1} = \overline{C_2}$.

Then the topological structure of X_1 = the topological structure of X_2.

In the sequel D_1 denotes a subset of X_1 and D_2 denotes a subset of X_2.
One can prove the following propositions:

(76) Suppose $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2. If D_1 is open, then D_2 is open.

(77) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\mathrm{Int}\, D_1 = \mathrm{Int}\, D_2$.

(78) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\mathrm{Int}\, D_1 \subseteq \mathrm{Int}\, D_2$.

(79) Suppose $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2. If D_1 is closed, then D_2 is closed.

(80) If $D_1 = D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\overline{D_1} = \overline{D_2}$.

(81) If $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2, then $\overline{D_1} \subseteq \overline{D_2}$.

(82) Suppose $D_2 \subseteq D_1$ and the topological structure of X_1 = the topological structure of X_2. If D_1 is boundary, then D_2 is boundary.

(83) Suppose $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2. If D_1 is dense, then D_2 is dense.

(84) Suppose $D_2 \subseteq D_1$ and the topological structure of X_1 = the topological structure of X_2. If D_1 is nowhere dense, then D_2 is nowhere dense.

In the sequel X_1, X_2 denote non empty topological spaces, D_1 denotes a subset of X_1, and D_2 denotes a subset of X_2.
We now state the proposition

(85) Suppose $D_1 \subseteq D_2$ and the topological structure of X_1 = the topological structure of X_2. If D_1 is everywhere dense, then D_2 is everywhere dense.

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