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Introduction to Trees

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Summary. The article consists of two parts: the first one deals with the concept of the prefixes of a finite sequence, the second one introduces and deals with the concept of tree. Besides some auxiliary propositions concerning finite sequences are presented. The trees are introduced as non-empty sets of finite sequences of natural numbers which are closed on prefixes and on sequences of less numbers (i.e. if $\langle n_1, n_2, \ldots, n_k \rangle$ is a vertex (element) of a tree and $m_i \leq n_i$ for $i = 1, 2, \ldots, k$, then $\langle m_1, m_2, \ldots, m_k \rangle$ also is). Finite trees, elementary trees with n leaves, the leaves and the subtrees of a tree, the inserting of a tree into another tree, with a node used for determining the place of insertion, antichains of prefixes, and height and width of finite trees are introduced.

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The articles [6], [8], [2], [7], [9], [4], [3], [5], and [1] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: D denotes a non empty set, X, x, y denote sets, k, n denote natural numbers, and p, q, r denote finite sequences of elements of $\mathbb{N}$.

The following propositions are true:

(1) For all finite sequences p, q such that $q = p \restriction \mathrm{Seg}\, n$ holds $\mathrm{len}\, q \leq n$.

(2) For all finite sequences p, q such that $q = p \restriction \mathrm{Seg}\, n$ holds $\mathrm{len}\, q \leq \mathrm{len}\, p$.

(3) For all finite sequences p, r such that $r = p \restriction \mathrm{Seg}\, n$ there exists a finite sequence q such that $p = r \frown q$.

(4) $\emptyset \neq \langle x \rangle$.

(5) For all finite sequences p, q such that $p = p \frown q$ or $p = q \frown p$ holds $q = \emptyset$.

(6) For all finite sequences p, q such that $p \frown q = \langle x \rangle$ holds $p = \langle x \rangle$ and $q = \emptyset$ or $p = \emptyset$ and $q = \langle x \rangle$.

Let p, q be finite sequences. Let us observe that $p \subseteq q$ if and only if:

(Def. 1) There exists n such that $p = q \restriction \mathrm{Seg}\, n$.

We introduce $p \preceq q$ as a synonym of $p \subseteq q$.

We now state three propositions:

(8)[1] For all finite sequences p, q holds $p \preceq q$ iff there exists a finite sequence r such that $q = p \frown r$.

(15)[2] For all finite sequences p, q such that $p \preceq q$ and $\mathrm{len}\, p = \mathrm{len}\, q$ holds $p = q$.

[1] The proposition (7) has been removed.
[2] The propositions (9)–(14) have been removed.

(16) $\langle x \rangle \preceq \langle y \rangle$ iff $x = y$.

Let p, q be finite sequences. We introduce $p \prec q$ as a synonym of $p \subset q$.
One can prove the following proposition

(19)[3] For all finite sets p, q such that p and q are $\subseteq$-comparable and $\operatorname{card} p = \operatorname{card} q$ holds $p = q$.

In the sequel p_1, p_2 denote finite sequences.
Next we state a number of propositions:

(23)[4] $\langle x \rangle$ and $\langle y \rangle$ are $\subseteq$-comparable iff $x = y$.

(24) For all finite sets p, q such that $p \subset q$ holds $\operatorname{card} p < \operatorname{card} q$.

(25) It is not true that there exists a finite sequence p such that $p \prec \emptyset$ or $p \prec \varepsilon_D$.

(26) It is not true that there exist finite sequences p, q such that $p \prec q$ and $q \prec p$.

(27) For all finite sequences p, q, r such that $p \prec q$ and $q \prec r$ or $p \prec q$ and $q \preceq r$ or $p \preceq q$ and $q \prec r$ holds $p \prec r$.

(28) If $p_1 \preceq p_2$, then $p_2 \not\prec p_1$.

(30)[5] If $p_1 \frown \langle x \rangle \preceq p_2$, then $p_1 \prec p_2$.

(31) If $p_1 \preceq p_2$, then $p_1 \prec p_2 \frown \langle x \rangle$.

(32) If $p_1 \prec p_2 \frown \langle x \rangle$, then $p_1 \preceq p_2$.

(33) If $\emptyset \prec p_2$ or $\emptyset \neq p_2$, then $p_1 \prec p_1 \frown p_2$.

Let p be a finite sequence. The functor $\operatorname{Seg}_{\preceq}(p)$ yields a set and is defined by:

(Def. 4)[6] $x \in \operatorname{Seg}_{\preceq}(p)$ iff there exists a finite sequence q such that $x = q$ and $q \prec p$.

Next we state several propositions:

(35)[7] For every finite sequence p such that $x \in \operatorname{Seg}_{\preceq}(p)$ holds x is a finite sequence.

(36) For all finite sequences p, q holds $p \in \operatorname{Seg}_{\preceq}(q)$ iff $p \prec q$.

(37) For all finite sequences p, q such that $p \in \operatorname{Seg}_{\preceq}(q)$ holds $\operatorname{len} p < \operatorname{len} q$.

(38) For all finite sequences p, q, r such that $q \frown r \in \operatorname{Seg}_{\preceq}(p)$ holds $q \in \operatorname{Seg}_{\preceq}(p)$.

(39) $\operatorname{Seg}_{\preceq}(\emptyset) = \emptyset$.

(40) $\operatorname{Seg}_{\preceq}(\langle x \rangle) = \{\emptyset\}$.

(41) For all finite sequences p, q such that $p \preceq q$ holds $\operatorname{Seg}_{\preceq}(p) \subseteq \operatorname{Seg}_{\preceq}(q)$.

(42) For all finite sequences p, q, r such that $q \in \operatorname{Seg}_{\preceq}(p)$ and $r \in \operatorname{Seg}_{\preceq}(p)$ holds q and r are $\subseteq$-comparable.

Let us consider X. We say that X is tree-like if and only if:

(Def. 5) $X \subseteq \mathbb{N}^*$ and for every p such that $p \in X$ holds $\operatorname{Seg}_{\preceq}(p) \subseteq X$ and for all p, k, n such that $p \frown \langle k \rangle \in X$ and $n \leq k$ holds $p \frown \langle n \rangle \in X$.

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[3] The propositions (17) and (18) have been removed.
[4] The propositions (20)–(22) have been removed.
[5] The proposition (29) has been removed.
[6] The definitions (Def. 2) and (Def. 3) have been removed.
[7] The proposition (34) has been removed.

One can check that there exists a set which is non empty and tree-like.
A tree is a tree-like non empty set.
In the sequel T, T_1 are trees.
The following proposition is true

(44)[8] If $x \in T$, then x is a finite sequence of elements of $\mathbb{N}$.

Let us consider T. We see that the element of T is a finite sequence of elements of $\mathbb{N}$.
The following propositions are true:

(45) For all finite sequences p, q such that $p \in T$ and $q \preceq p$ holds $q \in T$.

(46) For every finite sequence r such that $q \frown r \in T$ holds $q \in T$.

(47) $\emptyset \in T$ and $\varepsilon_{\mathbb{N}} \in T$.

(48) $\{\emptyset\}$ is a tree.

(49) $T \cup T_1$ is a tree.

(50) $T \cap T_1$ is a tree.

Let us note that there exists a tree which is finite.
In the sequel f_1, f_2 denote finite trees.
One can prove the following propositions:

(52)[9] $f_1 \cup f_2$ is a finite tree.

(53) $f_1 \cap T$ is a finite tree and $T \cap f_1$ is a finite tree.

Let us consider n. The elementary tree of n yielding a finite tree is defined as follows:

(Def. 7)[10] The elementary tree of $n = \{\langle k \rangle : k < n\} \cup \{\emptyset\}$.

Next we state three propositions:

(55)[11] If $k < n$, then $\langle k \rangle \in$ the elementary tree of n.

(56) The elementary tree of $0 = \{\emptyset\}$.

(57) If $p \in$ the elementary tree of n, then $p = \emptyset$ or there exists k such that $k < n$ and $p = \langle k \rangle$.

Let us consider T. The functor Leaves(T) yielding a subset of T is defined by:

(Def. 8) $p \in$ Leaves(T) iff $p \in T$ and it is not true that there exists q such that $q \in T$ and $p \prec q$.

Let us consider p. Let us assume that $p \in T$. The functor $T \restriction p$ yielding a tree is defined as follows:

(Def. 9) $q \in T \restriction p$ iff $p \frown q \in T$.

The following proposition is true

(60)[12] $T \restriction \varepsilon_{\mathbb{N}} = T$.

Let T be a finite tree and let p be an element of T. One can check that $T \restriction p$ is finite.
Let us consider T. Let us assume that Leaves$(T) \neq \emptyset$. An element of T is called a leaf of T if:

(Def. 10) It $\in$ Leaves(T).

[8] The proposition (43) has been removed.
[9] The proposition (51) has been removed.
[10] The definition (Def. 6) has been removed.
[11] The proposition (54) has been removed.
[12] The propositions (58) and (59) have been removed.

Let us consider T. A tree is called a subtree of T if:

(Def. 11) There exists an element p of T such that it $= T \upharpoonright p$.

In the sequel t denotes an element of T.

Let us consider T, p, T_1. Let us assume that $p \in T$. The functor T with-replacement(p, T_1) yielding a tree is defined as follows:

(Def. 12) $q \in T$ with-replacement(p, T_1) iff $q \in T$ and $p \not\preceq q$ or there exists r such that $r \in T_1$ and $q = p \frown r$.

We now state two propositions:

(64)[13] If $p \in T$, then T with-replacement$(p, T_1) = \{t_1; t_1$ ranges over elements of T: $p \not\preceq t_1\} \cup \{p \frown s; s$ ranges over elements of T_1: $s = s\}$.

(66)[14] If $p \in T$, then $T_1 = (T$ with-replacement$(p, T_1)) \upharpoonright p$.

Let T be a finite tree, let t be an element of T, and let T_1 be a finite tree. One can verify that T with-replacement(t, T_1) is finite.

In the sequel w denotes a finite sequence.

The following proposition is true

(67) For every finite sequence p holds $\mathrm{Seg}_{\preceq}(p) \approx \mathrm{dom}\, p$.

Let p be a finite sequence. Note that $\mathrm{Seg}_{\preceq}(p)$ is finite.

Next we state the proposition

(68) For every finite sequence p holds $\mathrm{card}\, \mathrm{Seg}_{\preceq}(p) = \mathrm{len}\, p$.

Let I_1 be a set. We say that I_1 is antichain of prefixes-like if and only if the conditions (Def. 13) are satisfied.

(Def. 13)(i) For every x such that $x \in I_1$ holds x is a finite sequence, and

(ii) for all p_1, p_2 such that $p_1 \in I_1$ and $p_2 \in I_1$ and $p_1 \neq p_2$ holds p_1 and p_2 are not $\subseteq$-comparable.

Let us mention that there exists a set which is antichain of prefixes-like.

An antichain of prefixes is an antichain of prefixes-like set.

Next we state two propositions:

(70)[15] $\{w\}$ is antichain of prefixes-like.

(71) If p_1 and p_2 are not $\subseteq$-comparable, then $\{p_1, p_2\}$ is antichain of prefixes-like.

Let us consider T. An antichain of prefixes is said to be an antichain of prefixes of T if:

(Def. 14) It $\subseteq T$.

In the sequel t_1, t_2 denote elements of T.

We now state three propositions:

(73)[16] $\emptyset$ is an antichain of prefixes of T and $\{\emptyset\}$ is an antichain of prefixes of T.

(74) $\{t\}$ is an antichain of prefixes of T.

(75) If t_1 and t_2 are not $\subseteq$-comparable, then $\{t_1, t_2\}$ is an antichain of prefixes of T.

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[13] The propositions (61)–(63) have been removed.

[14] The proposition (65) has been removed.

[15] The proposition (69) has been removed.

[16] The proposition (72) has been removed.

Let T be a finite tree. Observe that every antichain of prefixes of T is finite.

Let T be a finite tree. The functor height T yielding a natural number is defined as follows:

(Def. 15) There exists p such that $p \in T$ and $\operatorname{len} p = \operatorname{height} T$ and for every p such that $p \in T$ holds $\operatorname{len} p \leq \operatorname{height} T$.

The functor width T yields a natural number and is defined by:

(Def. 16) There exists an antichain X of prefixes of T such that $\operatorname{width} T = \operatorname{card} X$ and for every antichain Y of prefixes of T holds $\operatorname{card} Y \leq \operatorname{card} X$.

The following propositions are true:

(78)[17] $1 \leq \operatorname{width} f_1$.

(79) height (the elementary tree of 0) $= 0$.

(80) If $\operatorname{height} f_1 = 0$, then $f_1 =$ the elementary tree of 0.

(81) height (the elementary tree of $n+1$) $= 1$.

(82) width (the elementary tree of 0) $= 1$.

(83) width (the elementary tree of $n+1$) $= n+1$.

(84) For every element t of f_1 holds $\operatorname{height}(f_1 \upharpoonright t) \leq \operatorname{height} f_1$.

(85) For every element t of f_1 such that $t \neq \emptyset$ holds $\operatorname{height}(f_1 \upharpoonright t) < \operatorname{height} f_1$.

The scheme *Tree Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every f_1 holds $\mathcal{P}[f_1]$

provided the following requirement is met:

- For every f_1 such that for every n such that $\langle n \rangle \in f_1$ holds $\mathcal{P}[f_1 \upharpoonright \langle n \rangle]$ holds $\mathcal{P}[f_1]$.

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[17] The propositions (76) and (77) have been removed.