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Separated and Weakly Separated Subspaces of Topological Spaces

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Summary. A new concept of weakly separated subsets and subspaces of topological spaces is described in Mizar formalizm. Based on [1], in comparison with the notion of separated subsets (subspaces), some properties of such subsets (subspaces) are discussed. Some necessary facts concerning closed subspaces, open subspaces and the union and the meet of two subspaces are also introduced. To present the main theorems we first formulate basic definitions. Let X be a topological space. Two subsets A_1 and A_2 of X are called *weakly separated* if $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are separated. Two subspaces X_1 and X_2 of X are called *weakly separated* if their carriers are weakly separated. The following theorem contains a useful characterization of weakly separated subsets in the special case when $A_1 \cup A_2$ is equal to the carrier of X. *A_1 and A_2 are weakly separated iff there are such subsets of X, C_1 and C_2 closed (open) and C open (closed, respectively), that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$, $C_1 \subset A_1$, $C_2 \subset A_2$ and $C \subset A_1 \cap A_2$.* Next theorem divided into two parts contains similar characterization of weakly separated subspaces in the special case when the union of X_1 and X_2 is equal to X. *If X_1 meets X_2, then X_1 and X_2 are weakly separated iff either X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there are such open (closed) subspaces Y_1 and Y_2 of X that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 and either X is equal to the union of Y_1 and Y_2 or there is a(n) closed (open, respectively) subspace Y of X being a subspace of the meet of X_1 and X_2 and with the property that X is the union of all Y_1, Y_2 and Y. If X_1 misses X_2, then X_1 and X_2 are weakly separated iff X_1 and X_2 are open (closed) subspaces of X.* Moreover, the following simple characterization of separated subspaces by means of weakly separated ones is obtained. *X_1 and X_2 are separated iff there are weakly separated subspaces Y_1 and Y_2 of X such that X_1 is a subspace of Y_1, X_2 is a subspace of Y_2 and either Y_1 misses Y_2 or the meet of Y_1 and Y_2 misses the union of X_1 and X_2.*

MML Identifier: TSEP_1.

WWW: http://mizar.org/JFM/Vol4/tsep_1.html

The articles [4], [6], [3], [2], and [5] provide the notation and terminology for this paper.

1. SOME PROPERTIES OF SUBSPACES OF TOPOLOGICAL SPACES

In this paper X denotes a topological space.

We now state a number of propositions:

(1) For every topological structure X and for every subspace X_0 of X holds the carrier of X_0 is a subset of X.

(2) Every topological structure X is a subspace of X.

(3) For every strict topological structure X holds $X \restriction \Omega_X = X$.

(4) For all subspaces X_1, X_2 of X holds the carrier of $X_1 \subseteq$ the carrier of X_2 iff X_1 is a subspace of X_2.

(5) Let X be a topological structure and X_1, X_2 be subspaces of X. Suppose the carrier of $X_1 =$ the carrier of X_2. Then the topological structure of $X_1 =$ the topological structure of X_2.

(6) Let X_1, X_2 be subspaces of X. Suppose X_1 is a subspace of X_2 and X_2 is a subspace of X_1. Then the topological structure of $X_1 =$ the topological structure of X_2.

(7) For every subspace X_1 of X holds every subspace of X_1 is a subspace of X.

(8) Let X_0 be a subspace of X, C, A be subsets of X, and B be a subset of X_0. Suppose C is closed and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$. Then B is closed if and only if A is closed.

(9) Let X_0 be a subspace of X, C, A be subsets of X, and B be a subset of X_0. Suppose C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$. Then B is open if and only if A is open.

(10) Let X be a non empty topological structure and A_0 be a non empty subset of X. Then there exists a strict non empty subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(11) Let X_0 be a subspace of X and A be a subset of X. Suppose $A =$ the carrier of X_0. Then X_0 is a closed subspace of X if and only if A is closed.

(12) Let X_0 be a closed subspace of X, A be a subset of X, and B be a subset of X_0. If $A = B$, then B is closed iff A is closed.

(13) For every closed subspace X_1 of X holds every closed subspace of X_1 is a closed subspace of X.

(14) Let X be a non empty topological space, X_1 be a closed non empty subspace of X, and X_2 be a non empty subspace of X. Suppose the carrier of $X_1 \subseteq$ the carrier of X_2. Then X_1 is a closed non empty subspace of X_2.

(15) Let X be a non empty topological space and A_0 be a non empty subset of X. Suppose A_0 is closed. Then there exists a strict closed non empty subspace X_0 of X such that $A_0 =$ the carrier of X_0.

Let X be a topological structure and let I_1 be a subspace of X. We say that I_1 is open if and only if:

(Def. 1) For every subset A of X such that $A =$ the carrier of I_1 holds A is open.

Let X be a topological space. Observe that there exists a subspace of X which is strict and open.

Let X be a non empty topological space. Observe that there exists a subspace of X which is strict, open, and non empty.

One can prove the following propositions:

(16) Let X_0 be a subspace of X and A be a subset of X. Suppose $A =$ the carrier of X_0. Then X_0 is an open subspace of X if and only if A is open.

(17) Let X_0 be an open subspace of X, A be a subset of X, and B be a subset of X_0. If $A = B$, then B is open iff A is open.

(18) For every open subspace X_1 of X holds every open subspace of X_1 is an open subspace of X.

(19) Let X be a non empty topological space, X_1 be an open subspace of X, and X_2 be a non empty subspace of X. Suppose the carrier of $X_1 \subseteq$ the carrier of X_2. Then X_1 is an open subspace of X_2.

(20) Let X be a non empty topological space and A_0 be a non empty subset of X. Suppose A_0 is open. Then there exists a strict open non empty subspace X_0 of X such that $A_0 =$ the carrier of X_0.

2. Operations on subspaces of topological spaces

In the sequel X denotes a non empty topological space.

Let X be a non empty topological structure and let X_1, X_2 be non empty subspaces of X. The functor $X_1 \cup X_2$ yields a strict non empty subspace of X and is defined as follows:

(Def. 2) The carrier of $X_1 \cup X_2 =$ (the carrier of $X_1) \cup$ (the carrier of X_2).

Let us note that the functor $X_1 \cup X_2$ is commutative.

In the sequel X_1, X_2, X_3 are non empty subspaces of X.

We now state several propositions:

(21) $(X_1 \cup X_2) \cup X_3 = X_1 \cup (X_2 \cup X_3)$.

(22) X_1 is a subspace of $X_1 \cup X_2$.

(23) Let X_1, X_2 be non empty subspaces of X. Then X_1 is a subspace of X_2 if and only if $X_1 \cup X_2 =$ the topological structure of X_2.

(24) For all closed non empty subspaces X_1, X_2 of X holds $X_1 \cup X_2$ is a closed subspace of X.

(25) For all open non empty subspaces X_1, X_2 of X holds $X_1 \cup X_2$ is an open subspace of X.

Let X be a topological structure and let X_1, X_2 be subspaces of X. We say that X_1 misses X_2 if and only if:

(Def. 3) The carrier of X_1 misses the carrier of X_2.

Let us note that the predicate X_1 misses X_2 is symmetric. We introduce X_1 meets X_2 as an antonym of X_1 misses X_2.

Let X be a non empty topological structure and let X_1, X_2 be non empty subspaces of X. Let us assume that X_1 meets X_2. The functor $X_1 \cap X_2$ yielding a strict non empty subspace of X is defined by:

(Def. 5)[1] The carrier of $X_1 \cap X_2 =$ (the carrier of $X_1) \cap$ (the carrier of X_2).

In the sequel X_1, X_2, X_3 denote non empty subspaces of X.

We now state several propositions:

(29)[2] (i) If X_1 meets X_2, then $X_1 \cap X_2 = X_2 \cap X_1$, and

(ii) if X_1 meets X_2 and $X_1 \cap X_2$ meets X_3 or X_2 meets X_3 and X_1 meets $X_2 \cap X_3$, then $(X_1 \cap X_2) \cap X_3 = X_1 \cap (X_2 \cap X_3)$.

(30) If X_1 meets X_2, then $X_1 \cap X_2$ is a subspace of X_1 and $X_1 \cap X_2$ is a subspace of X_2.

(31) Let X_1, X_2 be non empty subspaces of X such that X_1 meets X_2. Then

(i) X_1 is a subspace of X_2 iff $X_1 \cap X_2 =$ the topological structure of X_1, and

(ii) X_2 is a subspace of X_1 iff $X_1 \cap X_2 =$ the topological structure of X_2.

(32) For all closed non empty subspaces X_1, X_2 of X such that X_1 meets X_2 holds $X_1 \cap X_2$ is a closed subspace of X.

(33) For all open non empty subspaces X_1, X_2 of X such that X_1 meets X_2 holds $X_1 \cap X_2$ is an open subspace of X.

(34) If X_1 meets X_2, then $X_1 \cap X_2$ is a subspace of $X_1 \cup X_2$.

(35) For every non empty subspace Y of X such that X_1 meets Y and Y meets X_2 holds $(X_1 \cup X_2) \cap Y = X_1 \cap Y \cup X_2 \cap Y$ and $Y \cap (X_1 \cup X_2) = Y \cap X_1 \cup Y \cap X_2$.

(36) For every non empty subspace Y of X such that X_1 meets X_2 holds $X_1 \cap X_2 \cup Y = (X_1 \cup Y) \cap (X_2 \cup Y)$ and $Y \cup X_1 \cap X_2 = (Y \cup X_1) \cap (Y \cup X_2)$.

[1] The definition (Def. 4) has been removed.

[2] The propositions (26)–(28) have been removed.

3. SEPARATED AND WEAKLY SEPARATED SUBSETS OF TOPOLOGICAL SPACES

In the sequel X denotes a topological space and A_1, A_2 denote subsets of X.

One can prove the following propositions:

(37) For all subsets A_1, A_2 of X such that A_1 and A_2 are separated holds A_1 misses A_2.

(38) If A_1 is closed and A_2 is closed, then A_1 misses A_2 iff A_1 and A_2 are separated.

(39) If $A_1 \cup A_2$ is closed and A_1 and A_2 are separated, then A_1 is closed and A_2 is closed.

(40) If A_1 misses A_2 and A_1 is open, then A_1 misses $\overline{A_2}$.

(41) If A_1 is open and A_2 is open, then A_1 misses A_2 iff A_1 and A_2 are separated.

(42) If $A_1 \cup A_2$ is open and A_1 and A_2 are separated, then A_1 is open and A_2 is open.

In the sequel A_1, A_2 are subsets of X.

We now state several propositions:

(43) For every subset C of X such that A_1 and A_2 are separated holds $A_1 \cap C$ and $A_2 \cap C$ are separated.

(44) Let B be a subset of X. Suppose A_1 and B are separated or A_2 and B are separated. Then $A_1 \cap A_2$ and B are separated.

(45) Let B be a subset of X. Then A_1 and B are separated and A_2 and B are separated if and only if $A_1 \cup A_2$ and B are separated.

(46) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and C_1 misses A_2 and C_2 misses A_1 and C_1 is closed and C_2 is closed.

(47) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and $C_1 \cap C_2$ misses $A_1 \cup A_2$ and C_1 is closed and C_2 is closed.

(48) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and C_1 misses A_2 and C_2 misses A_1 and C_1 is open and C_2 is open.

(49) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and $C_1 \cap C_2$ misses $A_1 \cup A_2$ and C_1 is open and C_2 is open.

Let X be a topological structure and let A_1, A_2 be subsets of X. We say that A_1 and A_2 are weakly separated if and only if:

(Def. 7)[3] $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are separated.

Let us note that the predicate A_1 and A_2 are weakly separated is symmetric.

In the sequel X denotes a topological space and A_1, A_2 denote subsets of X.

Next we state several propositions:

(51)[4] A_1 misses A_2 and A_1 and A_2 are weakly separated iff A_1 and A_2 are separated.

(52) If $A_1 \subseteq A_2$, then A_1 and A_2 are weakly separated.

(53) For all subsets A_1, A_2 of X such that A_1 is closed and A_2 is closed holds A_1 and A_2 are weakly separated.

(54) For all subsets A_1, A_2 of X such that A_1 is open and A_2 is open holds A_1 and A_2 are weakly separated.

[3] The definition (Def. 6) has been removed.

[4] The proposition (50) has been removed.

(55) For every subset C of X such that A_1 and A_2 are weakly separated holds $A_1 \cup C$ and $A_2 \cup C$ are weakly separated.

(56) Let B_1, B_2 be subsets of X. Suppose $B_1 \subseteq A_2$ and $B_2 \subseteq A_1$. Suppose A_1 and A_2 are weakly separated. Then $A_1 \cup B_1$ and $A_2 \cup B_2$ are weakly separated.

(57) Let B be a subset of X. Suppose A_1 and B are weakly separated and A_2 and B are weakly separated. Then $A_1 \cap A_2$ and B are weakly separated.

(58) Let B be a subset of X. Suppose A_1 and B are weakly separated and A_2 and B are weakly separated. Then $A_1 \cup A_2$ and B are weakly separated.

(59) A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$ and C_1 is closed and C_2 is closed and C is open.

In the sequel X denotes a non empty topological space and A_1, A_2 denote subsets of X.
One can prove the following propositions:

(60) Suppose A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non empty subsets C_1, C_2 of X such that
- (i) C_1 is closed,
- (ii) C_2 is closed,
- (iii) $C_1 \cap (A_1 \cup A_2) \subseteq A_1$,
- (iv) $C_2 \cap (A_1 \cup A_2) \subseteq A_2$, and
- (v) $A_1 \cup A_2 \subseteq C_1 \cup C_2$ or there exists a non empty subset C of X such that C is open and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$.

(61) Suppose $A_1 \cup A_2 =$ the carrier of X. Then A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C \subseteq A_1 \cap A_2$ and C_1 is closed and C_2 is closed and C is open.

(62) Suppose $A_1 \cup A_2 =$ the carrier of X and A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non empty subsets C_1, C_2 of X such that
- (i) C_1 is closed,
- (ii) C_2 is closed,
- (iii) $C_1 \subseteq A_1$,
- (iv) $C_2 \subseteq A_2$, and
- (v) $A_1 \cup A_2 = C_1 \cup C_2$ or there exists a non empty subset C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C \subseteq A_1 \cap A_2$ and C is open.

(63) A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$ and C_1 is open and C_2 is open and C is closed.

(64) Suppose A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non empty subsets C_1, C_2 of X such that
- (i) C_1 is open,
- (ii) C_2 is open,
- (iii) $C_1 \cap (A_1 \cup A_2) \subseteq A_1$,
- (iv) $C_2 \cap (A_1 \cup A_2) \subseteq A_2$, and
- (v) $A_1 \cup A_2 \subseteq C_1 \cup C_2$ or there exists a non empty subset C of X such that C is closed and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$.

(65) Suppose $A_1 \cup A_2 =$ the carrier of X. Then A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C \subseteq A_1 \cap A_2$ and C_1 is open and C_2 is open and C is closed.

(66) Suppose $A_1 \cup A_2 =$ the carrier of X and A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non empty subsets C_1, C_2 of X such that

(i) C_1 is open,

(ii) C_2 is open,

(iii) $C_1 \subseteq A_1$,

(iv) $C_2 \subseteq A_2$, and

(v) $A_1 \cup A_2 = C_1 \cup C_2$ or there exists a non empty subset C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C \subseteq A_1 \cap A_2$ and C is closed.

(67) A_1 and A_2 are separated if and only if there exist subsets B_1, B_2 of X such that B_1 and B_2 are weakly separated and $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ and $B_1 \cap B_2$ misses $A_1 \cup A_2$.

4. SEPARATED AND WEAKLY SEPARATED SUBSPACES OF TOPOLOGICAL SPACES

In the sequel X denotes a non empty topological space.

Let X be a topological structure and let X_1, X_2 be subspaces of X. We say that X_1 and X_2 are separated if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let A_1, A_2 be subsets of X. Suppose $A_1 =$ the carrier of X_1 and $A_2 =$ the carrier of X_2. Then A_1 and A_2 are separated.

Let us note that the predicate X_1 and X_2 are separated is symmetric.

In the sequel X_1, X_2 are non empty subspaces of X.

One can prove the following propositions:

(68) If X_1 and X_2 are separated, then X_1 misses X_2.

(70)[5] For all closed non empty subspaces X_1, X_2 of X holds X_1 misses X_2 iff X_1 and X_2 are separated.

(71) If $X = X_1 \cup X_2$ and X_1 and X_2 are separated, then X_1 is a closed subspace of X.

(72) If $X_1 \cup X_2$ is a closed subspace of X and X_1 and X_2 are separated, then X_1 is a closed subspace of X.

(73) For all open non empty subspaces X_1, X_2 of X holds X_1 misses X_2 iff X_1 and X_2 are separated.

(74) If $X = X_1 \cup X_2$ and X_1 and X_2 are separated, then X_1 is an open subspace of X.

(75) If $X_1 \cup X_2$ is an open subspace of X and X_1 and X_2 are separated, then X_1 is an open subspace of X.

(76) Let Y, X_1, X_2 be non empty subspaces of X. Suppose X_1 meets Y and X_2 meets Y. Suppose X_1 and X_2 are separated. Then $X_1 \cap Y$ and $X_2 \cap Y$ are separated and $Y \cap X_1$ and $Y \cap X_2$ are separated.

(77) Let Y_1, Y_2 be subspaces of X. Suppose Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2. If X_1 and X_2 are separated, then Y_1 and Y_2 are separated.

(78) Let Y be a non empty subspace of X. Suppose X_1 meets X_2. If X_1 and Y are separated, then $X_1 \cap X_2$ and Y are separated.

[5] The proposition (69) has been removed.

(79) Let Y be a non empty subspace of X. Then X_1 and Y are separated and X_2 and Y are separated if and only if $X_1 \cup X_2$ and Y are separated.

(80) X_1 and X_2 are separated if and only if there exist closed non empty subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 and Y_1 misses X_2 and Y_2 misses X_1.

(81) X_1 and X_2 are separated if and only if there exist closed non empty subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 but X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$.

(82) X_1 and X_2 are separated if and only if there exist open non empty subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 and Y_1 misses X_2 and Y_2 misses X_1.

(83) X_1 and X_2 are separated if and only if there exist open non empty subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 but X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$.

Let X be a topological structure and let X_1, X_2 be subspaces of X. We say that X_1 and X_2 are weakly separated if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let A_1, A_2 be subsets of X. Suppose $A_1 =$ the carrier of X_1 and $A_2 =$ the carrier of X_2. Then A_1 and A_2 are weakly separated.

Let us note that the predicate X_1 and X_2 are weakly separated is symmetric.

In the sequel X_1, X_2 are non empty subspaces of X.

We now state a number of propositions:

(85)[6] X_1 misses X_2 and X_1 and X_2 are weakly separated iff X_1 and X_2 are separated.

(86) If X_1 is a subspace of X_2, then X_1 and X_2 are weakly separated.

(87) For all closed subspaces X_1, X_2 of X holds X_1 and X_2 are weakly separated.

(88) For all open subspaces X_1, X_2 of X holds X_1 and X_2 are weakly separated.

(89) Let Y be a non empty subspace of X. Suppose X_1 and X_2 are weakly separated. Then $X_1 \cup Y$ and $X_2 \cup Y$ are weakly separated.

(90) Let Y_1, Y_2 be non empty subspaces of X. Suppose Y_1 is a subspace of X_2 and Y_2 is a subspace of X_1. Suppose X_1 and X_2 are weakly separated. Then $X_1 \cup Y_1$ and $X_2 \cup Y_2$ are weakly separated and $Y_1 \cup X_1$ and $Y_2 \cup X_2$ are weakly separated.

(91) Let Y, X_1, X_2 be non empty subspaces of X such that X_1 meets X_2. Then

(i) if X_1 and Y are weakly separated and X_2 and Y are weakly separated, then $X_1 \cap X_2$ and Y are weakly separated, and

(ii) if Y and X_1 are weakly separated and Y and X_2 are weakly separated, then Y and $X_1 \cap X_2$ are weakly separated.

(92) Let Y be a non empty subspace of X. Then

(i) if X_1 and Y are weakly separated and X_2 and Y are weakly separated, then $X_1 \cup X_2$ and Y are weakly separated, and

(ii) if Y and X_1 are weakly separated and Y and X_2 are weakly separated, then Y and $X_1 \cup X_2$ are weakly separated.

[6] The proposition (84) has been removed.

(93) Let X be a non empty topological space and X_1, X_2 be non empty subspaces of X. Suppose X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if one of the following conditions is satisfied:

(i) X_1 is a subspace of X_2, or

(ii) X_2 is a subspace of X_1, or

(iii) there exist closed non empty subspaces Y_1, Y_2 of X such that $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 but $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2 but $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$ or there exists an open non empty subspace Y of X such that the topological structure of $X = Y_1 \cup Y_2 \cup Y$ and $Y \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.

(94) Suppose $X = X_1 \cup X_2$ and X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if one of the following conditions is satisfied:

(i) X_1 is a subspace of X_2, or

(ii) X_2 is a subspace of X_1, or

(iii) there exist closed non empty subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_1 but Y_2 is a subspace of X_2 but $X = Y_1 \cup Y_2$ or there exists an open non empty subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and Y is a subspace of $X_1 \cap X_2$.

(95) Suppose $X = X_1 \cup X_2$ and X_1 misses X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is a closed subspace of X and X_2 is a closed subspace of X.

(96) Let X be a non empty topological space and X_1, X_2 be non empty subspaces of X. Suppose X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if one of the following conditions is satisfied:

(i) X_1 is a subspace of X_2, or

(ii) X_2 is a subspace of X_1, or

(iii) there exist open non empty subspaces Y_1, Y_2 of X such that $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 but $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2 but $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$ or there exists a closed non empty subspace Y of X such that the topological structure of $X = Y_1 \cup Y_2 \cup Y$ and $Y \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.

(97) Suppose $X = X_1 \cup X_2$ and X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if one of the following conditions is satisfied:

(i) X_1 is a subspace of X_2, or

(ii) X_2 is a subspace of X_1, or

(iii) there exist open non empty subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_1 but Y_2 is a subspace of X_2 but $X = Y_1 \cup Y_2$ or there exists a closed non empty subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and Y is a subspace of $X_1 \cap X_2$.

(98) Suppose $X = X_1 \cup X_2$ and X_1 misses X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is an open subspace of X and X_2 is an open subspace of X.

(99) X_1 and X_2 are separated if and only if there exist non empty subspaces Y_1, Y_2 of X such that Y_1 and Y_2 are weakly separated and X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 and Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$.

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