

On the Lattice of Subalgebras of a Universal Algebra

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The articles [10], [6], [12], [9], [13], [4], [5], [2], [7], [14], [1], [8], [11], and [3] provide the notation and terminology for this paper.

In this paper U_0 is a universal algebra and o is an operation of U_0 .

Let us consider U_0 . Family of subalgebras of U_0 is defined by:

(Def. 1) For every set U_1 such that $U_1 \in$ it holds U_1 is a subalgebra of U_0 .

Let us consider U_0 . Observe that there exists a family of subalgebras of U_0 which is non empty.

Let us consider U_0 . Then $\text{Sub}(U_0)$ is a non empty family of subalgebras of U_0 . Let U_2 be a non empty family of subalgebras of U_0 . We see that the element of U_2 is a subalgebra of U_0 .

Let us consider U_0 . Then $\sqcup_{(U_0)}$ is a binary operation on $\text{Sub}(U_0)$. Then $\sqcap_{(U_0)}$ is a binary operation on $\text{Sub}(U_0)$.

Let us consider U_0 and let u be an element of $\text{Sub}(U_0)$. The functor \bar{u} yielding a subset of U_0 is defined as follows:

(Def. 2) There exists a subalgebra U_1 of U_0 such that $u = U_1$ and $\bar{u} =$ the carrier of U_1 .

Let us consider U_0 . The functor $\text{Carr}(U_0)$ yielding a function from $\text{Sub}(U_0)$ into $2^{\text{the carrier of } U_0}$ is defined by:

(Def. 3) For every element u of $\text{Sub}(U_0)$ holds $(\text{Carr}(U_0))(u) = \bar{u}$.

Next we state several propositions:

- (1) For every set u holds $u \in \text{Sub}(U_0)$ iff there exists a strict subalgebra U_1 of U_0 such that $u = U_1$.
- (2) For every non empty subset H of U_0 and for every o such that $\text{arity } o = 0$ holds H is closed on o iff $o(\emptyset) \in H$.
- (3) For every subalgebra U_1 of U_0 holds the carrier of $U_1 \subseteq$ the carrier of U_0 .
- (4) For every non empty subset H of U_0 and for every o such that H is closed on o and $\text{arity } o = 0$ holds $o_H = o$.
- (5) $\text{Constants}(U_0) = \{o(\emptyset); o \text{ ranges over operation of } U_0: \text{arity } o = 0\}$.
- (6) For every universal algebra U_0 with constants and for every subalgebra U_1 of U_0 holds $\text{Constants}(U_0) = \text{Constants}(U_1)$.

Let U_0 be a universal algebra with constants. Note that every subalgebra of U_0 has constants. The following proposition is true

- (7) For every universal algebra U_0 with constants and for all subalgebras U_1, U_3 of U_0 holds $\text{Constants}(U_1) = \text{Constants}(U_3)$.

Let us consider U_0 . Then $\text{Carr}(U_0)$ is a function from $\text{Sub}(U_0)$ into $2^{\text{the carrier of } U_0}$ and it can be characterized by the condition:

- (Def. 4) For every element u of $\text{Sub}(U_0)$ and for every subalgebra U_1 of U_0 such that $u = U_1$ holds $(\text{Carr}(U_0))(u) = \text{the carrier of } U_1$.

The following propositions are true:

- (8) For every strict subalgebra H of U_0 and for every element u of U_0 holds $u \in (\text{Carr}(U_0))(H)$ iff $u \in H$.
- (9) For every non empty subset H of $\text{Sub}(U_0)$ holds $(\text{Carr}(U_0))^\circ H$ is non empty.
- (10) For every universal algebra U_0 with constants and for every strict subalgebra U_1 of U_0 holds $\text{Constants}(U_0) \subseteq (\text{Carr}(U_0))(U_1)$.
- (11) Let U_0 be a universal algebra with constants, U_1 be a subalgebra of U_0 , and a be a set. If a is an element of $\text{Constants}(U_0)$, then $a \in \text{the carrier of } U_1$.
- (12) For every universal algebra U_0 with constants and for every non empty subset H of $\text{Sub}(U_0)$ holds $\bigcap((\text{Carr}(U_0))^\circ H)$ is a non empty subset of U_0 .
- (13) For every universal algebra U_0 with constants holds the carrier of the lattice of subalgebras of $U_0 = \text{Sub}(U_0)$.
- (14) Let U_0 be a universal algebra with constants, H be a non empty subset of $\text{Sub}(U_0)$, and S be a non empty subset of U_0 . If $S = \bigcap((\text{Carr}(U_0))^\circ H)$, then S is operations closed.

Let U_0 be a strict universal algebra with constants and let H be a non empty subset of $\text{Sub}(U_0)$. The functor $\bigcap H$ yields a strict subalgebra of U_0 and is defined as follows:

- (Def. 5) The carrier of $\bigcap H = \bigcap((\text{Carr}(U_0))^\circ H)$.

The following propositions are true:

- (15) Let U_0 be a universal algebra with constants, l_1, l_2 be elements of the lattice of subalgebras of U_0 , and U_1, U_3 be strict subalgebras of U_0 . If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqcup l_2 = U_1 \sqcup U_3$.
- (16) Let U_0 be a universal algebra with constants, l_1, l_2 be elements of the lattice of subalgebras of U_0 , and U_1, U_3 be strict subalgebras of U_0 . If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqcap l_2 = U_1 \cap U_3$.
- (17) Let U_0 be a universal algebra with constants, l_1, l_2 be elements of the lattice of subalgebras of U_0 , and U_1, U_3 be strict subalgebras of U_0 . Suppose $l_1 = U_1$ and $l_2 = U_3$. Then $l_1 \sqsubseteq l_2$ if and only if the carrier of $U_1 \subseteq \text{the carrier of } U_3$.
- (18) Let U_0 be a universal algebra with constants, l_1, l_2 be elements of the lattice of subalgebras of U_0 , and U_1, U_3 be strict subalgebras of U_0 . If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqsubseteq l_2$ iff U_1 is a subalgebra of U_3 .
- (19) For every strict universal algebra U_0 with constants holds the lattice of subalgebras of U_0 is bounded.

Let U_0 be a strict universal algebra with constants. Observe that the lattice of subalgebras of U_0 is bounded.

Next we state several propositions:

- (20) For every strict universal algebra U_0 with constants and for every strict subalgebra U_1 of U_0 holds $\text{Gen}^{\text{UA}}(\text{Constants}(U_0)) \cap U_1 = \text{Gen}^{\text{UA}}(\text{Constants}(U_0))$.
- (21) For every strict universal algebra U_0 with constants holds $\perp_{\text{the lattice of subalgebras of } U_0} = \text{Gen}^{\text{UA}}(\text{Constants}(U_0))$.
- (22) Let U_0 be a strict universal algebra with constants, U_1 be a subalgebra of U_0 , and H be a subset of U_0 . If $H =$ the carrier of U_0 , then $\text{Gen}^{\text{UA}}(H) \sqcup U_1 = \text{Gen}^{\text{UA}}(H)$.
- (23) Let U_0 be a strict universal algebra with constants and H be a subset of U_0 . Suppose $H =$ the carrier of U_0 . Then $\top_{\text{the lattice of subalgebras of } U_0} = \text{Gen}^{\text{UA}}(H)$.
- (24) For every strict universal algebra U_0 with constants holds $\top_{\text{the lattice of subalgebras of } U_0} = U_0$.
- (25) For every strict universal algebra U_0 with constants holds the lattice of subalgebras of U_0 is complete.

Let U_4, U_5 be universal algebras with constants and let F be a function from the carrier of U_4 into the carrier of U_5 . The functor $\text{FuncLatt}(F)$ yielding a function from the carrier of the lattice of subalgebras of U_4 into the carrier of the lattice of subalgebras of U_5 is defined by:

- (Def. 6) For every strict subalgebra U_1 of U_4 and for every subset H of U_5 such that $H = F^\circ$ (the carrier of U_1) holds $(\text{FuncLatt}(F))(U_1) = \text{Gen}^{\text{UA}}(H)$.

One can prove the following proposition

- (26) Let U_0 be a strict universal algebra with constants and F be a function from the carrier of U_0 into the carrier of U_0 . Suppose $F = \text{id}_{\text{the carrier of } U_0}$. Then $\text{FuncLatt}(F) = \text{id}_{\text{the carrier of the lattice of subalgebras of } U_0}$.

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