

Quotient Vector Spaces and Functionals¹

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Summary. The article presents well known facts about quotient vector spaces and functionals (see [8]). There are repeated theorems and constructions with either weaker assumptions or in more general situations (see [11], [7], [10]). The construction of coefficient functionals and non degenerated functional in quotient vector space generated by functional in the given vector space are the only new things which are done.

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The articles [13], [5], [20], [12], [3], [1], [15], [2], [17], [7], [21], [4], [6], [14], [19], [18], [16], and [9] provide the notation and terminology for this paper.

1. AUXILIARY FACTS ABOUT DOUBLE LOOPS AND VECTOR SPACES

The following proposition is true

- (1) Let K be an add-associative right zeroed right complementable left distributive left unital non empty double loop structure and a be an element of K . Then $(-\mathbf{1}_K) \cdot a = -a$.

Let K be a double loop structure. The functor $\text{StructVectSp}(K)$ yields a strict vector space structure over K and is defined as follows:

(Def. 1) $\text{StructVectSp}(K) = \langle \text{the carrier of } K, \text{ the addition of } K, \text{ the zero of } K, \text{ the multiplication of } K \rangle$.

Let K be a non empty double loop structure. Observe that $\text{StructVectSp}(K)$ is non empty.

Let K be an Abelian non empty double loop structure. Observe that $\text{StructVectSp}(K)$ is Abelian.

Let K be an add-associative non empty double loop structure. Note that $\text{StructVectSp}(K)$ is add-associative.

Let K be a right zeroed non empty double loop structure. Note that $\text{StructVectSp}(K)$ is right zeroed.

Let K be a right complementable non empty double loop structure. Note that $\text{StructVectSp}(K)$ is right complementable.

Let K be an associative left unital distributive non empty double loop structure. Observe that $\text{StructVectSp}(K)$ is vector space-like.

Let K be a non degenerated non empty double loop structure. Observe that $\text{StructVectSp}(K)$ is non trivial.

Let K be a non degenerated non empty double loop structure. One can check that there exists a non empty vector space structure over K which is non trivial.

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Let K be an add-associative right zeroed right complementable non empty double loop structure. One can verify that there exists a non empty vector space structure over K which is add-associative, right zeroed, right complementable, and strict.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure. One can check that there exists a non empty vector space structure over K which is add-associative, right zeroed, right complementable, vector space-like, and strict.

Let K be an Abelian add-associative right zeroed right complementable associative left unital distributive non degenerated non empty double loop structure. Observe that there exists a non empty vector space structure over K which is Abelian, add-associative, right zeroed, right complementable, vector space-like, strict, and non trivial.

Next we state a number of propositions:

- (2) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, a be an element of K , V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K , and v be a vector of V . Then $0_K \cdot v = 0_V$ and $a \cdot 0_V = 0_V$.
- (3) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K , S, T be subspaces of V , and v be a vector of V . If $S \cap T = \mathbf{0}_V$ and $v \in S$ and $v \in T$, then $v = 0_V$.
- (4) Let K be a field, V be a vector space over K , x be a set, and v be a vector of V . Then $x \in \text{Lin}(\{v\})$ if and only if there exists an element a of K such that $x = a \cdot v$.
- (5) Let K be a field, V be a vector space over K , v be a vector of V , and a, b be scalars of V . If $v \neq 0_V$ and $a \cdot v = b \cdot v$, then $a = b$.
- (6) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K , and W_1, W_2 be subspaces of V . Suppose V is the direct sum of W_1 and W_2 . Let v, v_1, v_2 be vectors of V . If $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$, then $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.
- (7) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K , and W_1, W_2 be subspaces of V . Suppose V is the direct sum of W_1 and W_2 . Let v, v_1, v_2 be vectors of V . If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v = v_1 + v_2$.
- (8) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K , and W_1, W_2 be subspaces of V . Suppose V is the direct sum of W_1 and W_2 . Let v, v_1, v_2 be vectors of V . If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_1 \in W_1$ and $v_2 \in W_2$.
- (9) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K , and W_1, W_2 be subspaces of V . Suppose V is the direct sum of W_1 and W_2 . Let v, v_1, v_2 be vectors of V . If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_{\langle W_2, W_1 \rangle} = \langle v_2, v_1 \rangle$.
- (10) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K , and W_1, W_2 be subspaces of V . Suppose V is the direct sum of W_1 and W_2 . Let v be a vector of V . If $v \in W_1$, then $v_{\langle W_1, W_2 \rangle} = \langle v, 0_V \rangle$.
- (11) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K , and W_1, W_2 be subspaces of V . Suppose V is the direct sum of W_1 and W_2 . Let v be a vector of V . If $v \in W_2$, then $v_{\langle W_1, W_2 \rangle} = \langle 0_V, v \rangle$.

- (12) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K , V_1 be a subspace of V , W_1 be a subspace of V_1 , and v be a vector of V . If $v \in W_1$, then v is a vector of V_1 .
- (13) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K , V_1, V_2, W be subspaces of V , and W_1, W_2 be subspaces of W . If $W_1 = V_1$ and $W_2 = V_2$, then $W_1 + W_2 = V_1 + V_2$.
- (14) Let K be a field, V be a vector space over K , W be a subspace of V , v be a vector of V , and w be a vector of W . If $v = w$, then $\text{Lin}(\{w\}) = \text{Lin}(\{v\})$.
- (15) Let K be a field, V be a vector space over K , v be a vector of V , and X be a subspace of V . Suppose $v \notin X$. Let y be a vector of $X + \text{Lin}(\{v\})$ and W be a subspace of $X + \text{Lin}(\{v\})$. If $v = y$ and $W = X$, then $X + \text{Lin}(\{v\})$ is the direct sum of W and $\text{Lin}(\{y\})$.
- (16) Let K be a field, V be a vector space over K , v be a vector of V , X be a subspace of V , y be a vector of $X + \text{Lin}(\{v\})$, and W be a subspace of $X + \text{Lin}(\{v\})$. If $v = y$ and $X = W$ and $v \notin X$, then $y_{\langle W, \text{Lin}(\{y\}) \rangle} = \langle 0_W, y \rangle$.
- (17) Let K be a field, V be a vector space over K , v be a vector of V , X be a subspace of V , y be a vector of $X + \text{Lin}(\{v\})$, and W be a subspace of $X + \text{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \text{Lin}(\{v\})$. If $w \in X$, then $w_{\langle W, \text{Lin}(\{y\}) \rangle} = \langle w, 0_V \rangle$.
- (18) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K , v be a vector of V , and W_1, W_2 be subspaces of V . Then there exist vectors v_1, v_2 of V such that $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.
- (19) Let K be a field, V be a vector space over K , v be a vector of V , X be a subspace of V , y be a vector of $X + \text{Lin}(\{v\})$, and W be a subspace of $X + \text{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \text{Lin}(\{v\})$. Then there exists a vector x of X and there exists an element r of K such that $w_{\langle W, \text{Lin}(\{y\}) \rangle} = \langle x, r \cdot v \rangle$.
- (20) Let K be a field, V be a vector space over K , v be a vector of V , X be a subspace of V , y be a vector of $X + \text{Lin}(\{v\})$, and W be a subspace of $X + \text{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w_1, w_2 be vectors of $X + \text{Lin}(\{v\})$, x_1, x_2 be vectors of X , and r_1, r_2 be elements of K . If $(w_1)_{\langle W, \text{Lin}(\{y\}) \rangle} = \langle x_1, r_1 \cdot v \rangle$ and $(w_2)_{\langle W, \text{Lin}(\{y\}) \rangle} = \langle x_2, r_2 \cdot v \rangle$, then $(w_1 + w_2)_{\langle W, \text{Lin}(\{y\}) \rangle} = \langle x_1 + x_2, (r_1 + r_2) \cdot v \rangle$.
- (21) Let K be a field, V be a vector space over K , v be a vector of V , X be a subspace of V , y be a vector of $X + \text{Lin}(\{v\})$, and W be a subspace of $X + \text{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \text{Lin}(\{v\})$, x be a vector of X , and t, r be elements of K . If $w_{\langle W, \text{Lin}(\{y\}) \rangle} = \langle x, r \cdot v \rangle$, then $(t \cdot w)_{\langle W, \text{Lin}(\{y\}) \rangle} = \langle t \cdot x, t \cdot r \cdot v \rangle$.

2. QUOTIENT VECTOR SPACE FOR NON COMMUTATIVE DOUBLE LOOP

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K , and let W be a subspace of V . The functor $\text{CosetSet}(V, W)$ yields a non empty family of subsets of V and is defined by:

(Def. 2) $\text{CosetSet}(V, W) = \{A : A \text{ ranges over cosets of } W\}$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K , and let W be a subspace of V . The functor $\text{addCoset}(V, W)$ yields a binary operation on $\text{CosetSet}(V, W)$ and is defined as follows:

(Def. 3) For all elements A, B of $\text{CosetSet}(V, W)$ and for all vectors a, b of V such that $A = a + W$ and $B = b + W$ holds $(\text{addCoset}(V, W))(A, B) = a + b + W$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K , and let W be a subspace of V . The functor $\text{zeroCoset}(V, W)$ yielding an element of $\text{CosetSet}(V, W)$ is defined by:

(Def. 4) $\text{zeroCoset}(V, W) =$ the carrier of W .

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K , and let W be a subspace of V . The functor $\text{ImultCoset}(V, W)$ yielding a function from [the carrier of K , $\text{CosetSet}(V, W)$] into $\text{CosetSet}(V, W)$ is defined by the condition (Def. 5).

(Def. 5) Let z be an element of K , A be an element of $\text{CosetSet}(V, W)$, and a be a vector of V . If $A = a + W$, then $(\text{ImultCoset}(V, W))(z, A) = z \cdot a + W$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K , and let W be a subspace of V . The functor $^V/W$ yielding a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over K is defined by the conditions (Def. 6).

- (Def. 6)(i) The carrier of $^V/W = \text{CosetSet}(V, W)$,
- (ii) the addition of $^V/W = \text{addCoset}(V, W)$,
- (iii) the zero of $^V/W = \text{zeroCoset}(V, W)$, and
- (iv) the left multiplication of $^V/W = \text{ImultCoset}(V, W)$.

We now state several propositions:

- (22) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K , and W be a subspace of V . Then $\text{zeroCoset}(V, W) = 0_V + W$ and $0_{^V/W} = \text{zeroCoset}(V, W)$.
- (23) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K , W be a subspace of V , and w be a vector of $^V/W$. Then w is a coset of W and there exists a vector v of V such that $w = v + W$.
- (24) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K , W be a subspace of V , and v be a vector of V . Then $v + W$ is a coset of W and $v + W$ is a vector of $^V/W$.
- (25) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K , and W be a subspace of V . Then every coset of W is a vector of $^V/W$.
- (26) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K , W be a subspace of V , A be a vector of $^V/W$, v be a vector of V , and a be a scalar of V . If $A = v + W$, then $a \cdot A = a \cdot v + W$.
- (27) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K , W be a subspace of V , A_1, A_2 be vectors of $^V/W$, and v_1, v_2 be vectors of V . If $A_1 = v_1 + W$ and $A_2 = v_2 + W$, then $A_1 + A_2 = v_1 + v_2 + W$.

3. AUXILIARY FACTS ABOUT FUNCTIONALS

The following proposition is true

- (28) Let K be a field, V be a vector space over K , X be a subspace of V , f_1 be a linear functional in X , v be a vector of V , and y be a vector of $X + \text{Lin}(\{v\})$. Suppose $v = y$ and $v \notin X$. Let r be an element of K . Then there exists a linear functional p_1 in $X + \text{Lin}(\{v\})$ such that $p_1 \upharpoonright$ the carrier of $X = f_1$ and $p_1(y) = r$.

Let K be a right zeroed non empty loop structure and let V be a non empty vector space structure over K . One can check that there exists a functional in V which is additive and 0-preserving.

Let K be an add-associative right zeroed right complementable non empty double loop structure and let V be a right zeroed non empty vector space structure over K . Observe that every functional in V which is additive is also 0-preserving.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure and let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K . One can check that every functional in V which is homogeneous is also 0-preserving.

Let K be a non empty zero structure and let V be a non empty vector space structure over K . Observe that $0\text{Functional}V$ is constant.

Let K be a non empty zero structure and let V be a non empty vector space structure over K . Note that there exists a functional in V which is constant.

Let K be an add-associative right zeroed right complementable non empty double loop structure, let V be a right zeroed non empty vector space structure over K , and let f be a 0-preserving functional in V . Let us observe that f is constant if and only if:

(Def. 7) $f = 0\text{Functional}V$.

Let K be an add-associative right zeroed right complementable non empty double loop structure and let V be a right zeroed non empty vector space structure over K . Observe that there exists a functional in V which is constant, additive, and 0-preserving.

Let K be a non empty 1-sorted structure and let V be a non empty vector space structure over K . Observe that every functional in V which is non constant is also non trivial.

Let K be a field and let V be a non trivial vector space over K . Observe that there exists a functional in V which is additive, homogeneous, non constant, and non trivial.

Let K be a field and let V be a non trivial vector space over K . One can check that every functional in V which is trivial is also constant.

Let K be a field, let V be a non trivial vector space over K , let v be a vector of V , and let W be a linear complement of $\text{Lin}(\{v\})$. Let us assume that $v \neq 0_V$. The functor $\text{coeffFunctional}(v, W)$ yields a non constant non trivial linear functional in V and is defined by:

(Def. 8) $(\text{coeffFunctional}(v, W))(v) = \mathbf{1}_K$ and $\text{coeffFunctional}(v, W) \upharpoonright$ the carrier of $W = 0\text{Functional}W$.

One can prove the following propositions:

- (29) Let K be a field, V be a non trivial vector space over K , and f be a non constant 0-preserving functional in V . Then there exists a vector v of V such that $v \neq 0_V$ and $f(v) \neq 0_K$.
- (30) Let K be a field, V be a non trivial vector space over K , v be a vector of V , a be a scalar of V , and W be a linear complement of $\text{Lin}(\{v\})$. If $v \neq 0_V$, then $(\text{coeffFunctional}(v, W))(a \cdot v) = a$.
- (31) Let K be a field, V be a non trivial vector space over K , v, w be vectors of V , and W be a linear complement of $\text{Lin}(\{v\})$. If $v \neq 0_V$ and $w \in W$, then $(\text{coeffFunctional}(v, W))(w) = 0_K$.
- (32) Let K be a field, V be a non trivial vector space over K , v, w be vectors of V , a be a scalar of V , and W be a linear complement of $\text{Lin}(\{v\})$. If $v \neq 0_V$ and $w \in W$, then $(\text{coeffFunctional}(v, W))(a \cdot v + w) = a$.
- (33) Let K be a non empty loop structure, V be a non empty vector space structure over K , f, g be functionals in V , and v be a vector of V . Then $(f - g)(v) = f(v) - g(v)$.

Let K be a field and let V be a non trivial vector space over K . One can verify that \overline{V} is non trivial.

4. KERNEL OF ADDITIVE FUNCTIONAL. LINEAR FUNCTIONALS IN QUOTIENT VECTOR SPACES

Let K be a non empty zero structure, let V be a non empty vector space structure over K , and let f be a functional in V . The functor $\ker f$ yielding a subset of V is defined by:

(Def. 9) $\ker f = \{v; v \text{ ranges over vectors of } V: f(v) = 0_K\}$.

Let K be a right zeroed non empty loop structure, let V be a non empty vector space structure over K , and let f be a 0-preserving functional in V . Note that $\ker f$ is non empty.

One can prove the following proposition

(34) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K , and f be a linear functional in V . Then $\ker f$ is linearly closed.

Let K be a non empty zero structure, let V be a non empty vector space structure over K , and let f be a functional in V . We say that f is degenerated if and only if:

(Def. 10) $\ker f \neq \{0_V\}$.

Let K be a non degenerated non empty double loop structure and let V be a non trivial non empty vector space structure over K . Note that every functional in V which is non degenerated and 0-preserving is also non constant.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K , and let f be a linear functional in V . The functor $\text{Ker } f$ yields a strict non empty subspace of V and is defined by:

(Def. 11) The carrier of $\text{Ker } f = \ker f$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K , let W be a subspace of V , and let f be an additive functional in V . Let us assume that the carrier of $W \subseteq \ker f$. The functor f/W yielding an additive functional in V/W is defined as follows:

(Def. 12) For every vector A of V/W and for every vector a of V such that $A = a + W$ holds $(f/W)(A) = f(a)$.

One can prove the following proposition

(35) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K , W be a subspace of V , and f be a linear functional in V . If the carrier of $W \subseteq \ker f$, then f/W is homogeneous.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K , and let f be a linear functional in V . The functor $\text{CQFunctional } f$ yielding a linear functional in $V/\text{Ker } f$ is defined by:

(Def. 13) $\text{CQFunctional } f = f/\text{Ker } f$.

One can prove the following proposition

(36) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K , f be a linear functional in V , A be a vector of $V/\text{Ker } f$, and v be a vector of V . If $A = v + \text{Ker } f$, then $\text{CQFunctional } f(A) = f(v)$.

Let K be a field, let V be a non trivial vector space over K , and let f be a non constant linear functional in V . One can verify that $\text{CQFunctional } f$ is non constant.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K , and let f be a linear functional in V . Note that $\text{CQFunctional } f$ is non degenerated.

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