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The Well Ordering Relations

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Summary. Some theorems about well ordering relations are proved. The goal of the article is to prove that every two well ordering relations are either isomorphic or one of them is isomorphic to a segment of the other. The following concepts are defined: the segment of a relation induced by an element, well founded relations, well ordering relations, the restriction of a relation to a set, and the isomorphism of two relations. A number of simple facts is presented.

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The articles [3], [2], [4], [5], and [1] provide the notation and terminology for this paper.

We adopt the following convention: a, b, c, x, X, Y, Z are sets and R, S, T are binary relations.

Let us consider R, a. The functor $R\text{-Seg}(a)$ yielding a set is defined as follows:

(Def. 1) $x \in R\text{-Seg}(a)$ iff $x \neq a$ and $\langle x, a \rangle \in R$.

The following proposition is true

(2)[1] $x \in \text{field}\,R$ or $R\text{-Seg}(x) = \emptyset$.

Let us consider R. We say that R is well founded if and only if:

(Def. 2) For every Y such that $Y \subseteq \text{field}\,R$ and $Y \neq \emptyset$ there exists a such that $a \in Y$ and $R\text{-Seg}(a)$ misses Y.

Let us consider X. We say that R is well founded in X if and only if:

(Def. 3) For every Y such that $Y \subseteq X$ and $Y \neq \emptyset$ there exists a such that $a \in Y$ and $R\text{-Seg}(a)$ misses Y.

Next we state the proposition

(5)[2] R is well founded iff R is well founded in $\text{field}\,R$.

Let us consider R. We say that R is well-ordering if and only if:

(Def. 4) R is reflexive, transitive, antisymmetric, connected, and well founded.

Let us consider X. We say that R well orders X if and only if:

(Def. 5) R is reflexive in X, transitive in X, antisymmetric in X, connected in X, and well founded in X.

[1] The proposition (1) has been removed.
[2] The propositions (3) and (4) have been removed.

One can prove the following propositions:

(8)[3] R well orders field R iff R is well-ordering.

(9) Suppose R well orders X. Let given Y. Suppose $Y \subseteq X$ and $Y \neq \emptyset$. Then there exists a such that $a \in Y$ and for every b such that $b \in Y$ holds $\langle a, b \rangle \in R$.

(10) Suppose R is well-ordering. Let given Y. Suppose $Y \subseteq$ field R and $Y \neq \emptyset$. Then there exists a such that $a \in Y$ and for every b such that $b \in Y$ holds $\langle a, b \rangle \in R$.

(11) For every R such that R is well-ordering and field $R \neq \emptyset$ there exists a such that $a \in$ field R and for every b such that $b \in$ field R holds $\langle a, b \rangle \in R$.

(12) Let given R. Suppose R is well-ordering and field $R \neq \emptyset$. Let given a. Suppose $a \in$ field R. Then

(i) for every b such that $b \in$ field R holds $\langle b, a \rangle \in R$, or

(ii) there exists b such that $b \in$ field R and $\langle a, b \rangle \in R$ and for every c such that $c \in$ field R and $\langle a, c \rangle \in R$ holds $c = a$ or $\langle b, c \rangle \in R$.

In the sequel F, G denote functions.
Next we state the proposition

(13) R-Seg$(a) \subseteq$ field R.

Let us consider R, Y. The functor $R|^2 Y$ yielding a binary relation is defined as follows:

(Def. 6) $R|^2 Y = R \cap [:Y, Y:]$.

One can prove the following propositions:

(15)[4] $R|^2 X \subseteq R$ and $R|^2 X \subseteq [:X, X:]$.

(16) $x \in R|^2 X$ iff $x \in R$ and $x \in [:X, X:]$.

(17) $R|^2 X = X \upharpoonright R \upharpoonright X$.

(18) $R|^2 X = X \upharpoonright (R \upharpoonright X)$.

(19) If $x \in$ field$(R|^2 X)$, then $x \in$ field R and $x \in X$.

(20) field$(R|^2 X) \subseteq$ field R and field$(R|^2 X) \subseteq X$.

(21) $(R|^2 X)$-Seg$(a) \subseteq R$-Seg(a).

(22) If R is reflexive, then $R|^2 X$ is reflexive.

(23) If R is connected, then $R|^2 Y$ is connected.

(24) If R is transitive, then $R|^2 Y$ is transitive.

(25) If R is antisymmetric, then $R|^2 Y$ is antisymmetric.

(26) $R|^2 X|^2 Y = R|^2 (X \cap Y)$.

(27) $R|^2 X|^2 Y = R|^2 Y|^2 X$.

(28) $R|^2 Y|^2 Y = R|^2 Y$.

(29) If $Z \subseteq Y$, then $R|^2 Y|^2 Z = R|^2 Z$.

(30) $R|^2$ field $R = R$.

[3] The propositions (6) and (7) have been removed.
[4] The proposition (14) has been removed.

(31) If R is well founded, then $R\,|^2\,X$ is well founded.

(32) If R is well-ordering, then $R\,|^2\,Y$ is well-ordering.

(33) If R is well-ordering, then R-Seg(a) and R-Seg(b) are $\subseteq$-comparable.

(35)[5] If R is well-ordering and $a \in \mathrm{field}\,R$ and $b \in R$-Seg(a), then $(R\,|^2\,R$-Seg$(a))$-Seg$(b) =$ R-Seg(b).

(36) Suppose R is well-ordering and $Y \subseteq \mathrm{field}\,R$. Then $Y = \mathrm{field}\,R$ or there exists a such that $a \in \mathrm{field}\,R$ and $Y = R$-Seg(a) if and only if for every a such that $a \in Y$ and for every b such that $\langle b, a\rangle \in R$ holds $b \in Y$.

(37) If R is well-ordering and $a \in \mathrm{field}\,R$ and $b \in \mathrm{field}\,R$, then $\langle a, b\rangle \in R$ iff R-Seg$(a) \subseteq R$-Seg(b).

(38) If R is well-ordering and $a \in \mathrm{field}\,R$ and $b \in \mathrm{field}\,R$, then R-Seg$(a) \subseteq R$-Seg(b) iff $a = b$ or $a \in R$-Seg(b).

(39) If R is well-ordering and $X \subseteq \mathrm{field}\,R$, then $\mathrm{field}(R\,|^2\,X) = X$.

(40) If R is well-ordering, then $\mathrm{field}(R\,|^2\,R$-Seg$(a)) = R$-Seg(a).

(41) If R is well-ordering, then for every Z such that for every a such that $a \in \mathrm{field}\,R$ and R-Seg$(a) \subseteq Z$ holds $a \in Z$ holds $\mathrm{field}\,R \subseteq Z$.

(42) If R is well-ordering and $a \in \mathrm{field}\,R$ and $b \in \mathrm{field}\,R$ and for every c such that $c \in R$-Seg(a) holds $\langle c, b\rangle \in R$ and $c \neq b$, then $\langle a, b\rangle \in R$.

(43) Suppose R is well-ordering and $\mathrm{dom}\,F = \mathrm{field}\,R$ and $\mathrm{rng}\,F \subseteq \mathrm{field}\,R$ and for all a, b such that $\langle a, b\rangle \in R$ and $a \neq b$ holds $\langle F(a), F(b)\rangle \in R$ and $F(a) \neq F(b)$. Let given a. If $a \in \mathrm{field}\,R$, then $\langle a, F(a)\rangle \in R$.

Let us consider R, S, F. We say that F is an isomorphism between R and S if and only if:

(Def. 7) $\mathrm{dom}\,F = \mathrm{field}\,R$ and $\mathrm{rng}\,F = \mathrm{field}\,S$ and F is one-to-one and for all a, b holds $\langle a, b\rangle \in R$ iff $a \in \mathrm{field}\,R$ and $b \in \mathrm{field}\,R$ and $\langle F(a), F(b)\rangle \in S$.

The following proposition is true

(45)[6] If F is an isomorphism between R and S, then for all a, b such that $\langle a, b\rangle \in R$ and $a \neq b$ holds $\langle F(a), F(b)\rangle \in S$ and $F(a) \neq F(b)$.

Let us consider R, S. We say that R and S are isomorphic if and only if:

(Def. 8) There exists F which is an isomorphism between R and S.

The following propositions are true:

(47)[7] $\mathrm{id}_{\mathrm{field}\,R}$ is an isomorphism between R and R.

(48) R and R are isomorphic.

(49) If F is an isomorphism between R and S, then F^{-1} is an isomorphism between S and R.

(50) If R and S are isomorphic, then S and R are isomorphic.

(51) Suppose F is an isomorphism between R and S and G is an isomorphism between S and T. Then $G \cdot F$ is an isomorphism between R and T.

(52) If R and S are isomorphic and S and T are isomorphic, then R and T are isomorphic.

[5] The proposition (34) has been removed.
[6] The proposition (44) has been removed.
[7] The proposition (46) has been removed.

(53) Suppose F is an isomorphism between R and S. Then

(i) if R is reflexive, then S is reflexive,

(ii) if R is transitive, then S is transitive,

(iii) if R is connected, then S is connected,

(iv) if R is antisymmetric, then S is antisymmetric, and

(v) if R is well founded, then S is well founded.

(54) If R is well-ordering and F is an isomorphism between R and S, then S is well-ordering.

(55) Suppose R is well-ordering. Let given F, G. Suppose F is an isomorphism between R and S and G is an isomorphism between R and S. Then $F = G$.

Let us consider R, S. Let us assume that R is well-ordering and R and S are isomorphic. The canonical isomorphism between R and S yielding a function is defined by:

(Def. 9) The canonical isomorphism between R and S is an isomorphism between R and S.

We now state several propositions:

(57)[8] If R is well-ordering, then for every a such that $a \in \mathrm{field}\, R$ holds R and $R\,|^2\, R\text{-Seg}(a)$ are not isomorphic.

(58) If R is well-ordering and $a \in \mathrm{field}\, R$ and $b \in \mathrm{field}\, R$ and $a \neq b$, then $R\,|^2\, R\text{-Seg}(a)$ and $R\,|^2\, R\text{-Seg}(b)$ are not isomorphic.

(59) Suppose R is well-ordering and $Z \subseteq \mathrm{field}\, R$ and F is an isomorphism between R and S. Then $F \restriction Z$ is an isomorphism between $R\,|^2\, Z$ and $S\,|^2\, F^\circ Z$ and $R\,|^2\, Z$ and $S\,|^2\, F^\circ Z$ are isomorphic.

(60) Suppose R is well-ordering and F is an isomorphism between R and S. Let given a. If $a \in \mathrm{field}\, R$, then there exists b such that $b \in \mathrm{field}\, S$ and $F^\circ R\text{-Seg}(a) = S\text{-Seg}(b)$.

(61) Suppose R is well-ordering and F is an isomorphism between R and S. Let given a. If $a \in \mathrm{field}\, R$, then there exists b such that $b \in \mathrm{field}\, S$ and $R\,|^2\, R\text{-Seg}(a)$ and $S\,|^2\, S\text{-Seg}(b)$ are isomorphic.

(62) Suppose that R is well-ordering and S is well-ordering and $a \in \mathrm{field}\, R$ and $b \in \mathrm{field}\, S$ and $c \in \mathrm{field}\, S$ and R and $S\,|^2\, S\text{-Seg}(b)$ are isomorphic and $R\,|^2\, R\text{-Seg}(a)$ and $S\,|^2\, S\text{-Seg}(c)$ are isomorphic. Then $S\text{-Seg}(c) \subseteq S\text{-Seg}(b)$ and $\langle c, b\rangle \in S$.

(63) Suppose R is well-ordering and S is well-ordering. Then

(i) R and S are isomorphic, or

(ii) there exists a such that $a \in \mathrm{field}\, R$ and $R\,|^2\, R\text{-Seg}(a)$ and S are isomorphic, or

(iii) there exists a such that $a \in \mathrm{field}\, S$ and R and $S\,|^2\, S\text{-Seg}(a)$ are isomorphic.

(64) Suppose $Y \subseteq \mathrm{field}\, R$ and R is well-ordering. Then R and $R\,|^2\, Y$ are isomorphic or there exists a such that $a \in \mathrm{field}\, R$ and $R\,|^2\, R\text{-Seg}(a)$ and $R\,|^2\, Y$ are isomorphic.

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[8] The proposition (56) has been removed.

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