

Axioms of Incidency

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Summary. This article is based on “*Foundations of Geometry*” by Karol Borsuk and Wanda Szmielew ([1]). The fourth axiom of incidence is weakened. In [1] it has the form *for any plane there exist three non-collinear points in the plane* and in the article *for any plane there exists one point in the plane*. The original axiom is proved. The article includes: theorems concerning collinearity of points and coplanarity of points and lines, basic theorems concerning lines and planes, fundamental existence theorems, theorems concerning intersection of lines and planes.

The articles [3], [2], and [4] provide the terminology and notation for this paper. We consider structures IncStruct, which are systems

$$\langle\langle \text{Points, Lines, Planes, Inc}_1, \text{Inc}_2, \text{Inc}_3 \rangle\rangle$$

where Points, Lines, Planes have the type DOMAIN, Inc₁ has the type Relation **of the** Points, **the** Lines, Inc₂ has the type Relation **of the** Points, **the** Planes, and Inc₃ has the type Relation **of the** Lines, **the** Planes. We now define three new modes. Let S have the type IncStruct.

POINT **of** S stands for Element **of the** Points **of** S .

LINE **of** S stands for Element **of the** Lines **of** S .

PLANE **of** S stands for Element **of the** Planes **of** S .

In the sequel S will have the type IncStruct; A will have the type Element **of the** Points **of** S ; L will have the type Element **of the** Lines **of** S ; P will have the type Element **of the** Planes **of** S . The following propositions are true:

- (1) A is POINT **of** S ,
- (2) L is LINE **of** S ,

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(3) P is PLANE of S .

For simplicity we adopt the following convention: A, B, C, D will denote objects of the type POINT of S ; L will denote an object of the type LINE of S ; P will denote an object of the type PLANE of S ; F, G will denote objects of the type Subset of the Points of S . The arguments of the notions defined below are the following: S which is an object of the type reserved above; A which is an object of the type POINT of S ; L which is an object of the type LINE of S . The predicate

A on L is defined by $\langle A, L \rangle \in \mathbf{the\ Inc}_1$ of S .

The arguments of the notions defined below are the following: S which is an object of the type reserved above; A which is an object of the type POINT of S ; P which is an object of the type PLANE of S . The predicate

A on P is defined by $\langle A, P \rangle \in \mathbf{the\ Inc}_2$ of S .

The arguments of the notions defined below are the following: S which is an object of the type reserved above; L which is an object of the type LINE of S ; P which is an object of the type PLANE of S . The predicate

L on P is defined by $\langle L, P \rangle \in \mathbf{the\ Inc}_3$ of S .

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type set of POINT of S ; L which is an object of the type LINE of S . The predicate

F on L is defined by **for A being POINT of S st $A \in F$ holds A on L .**

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type set of POINT of S ; P which is an object of the type PLANE of S . The predicate

F on P is defined by **for A st $A \in F$ holds A on P .**

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type set of POINT of S . The predicate

F is_linear is defined by **ex L st F on L .**

The arguments of the notions defined below are the following: S which is an object of the type reserved above; F which is an object of the type set of POINT of S . The predicate

F is_planar is defined by **ex P st F on P .**

Next we state a number of propositions:

(4) A on L iff $\langle A, L \rangle \in \mathbf{the\ Inc}_1$ of S ,

- (5) A on P iff $\langle A, P \rangle \in \mathbf{the\ Inc}_2$ of S ,
- (6) L on P iff $\langle L, P \rangle \in \mathbf{the\ Inc}_3$ of S ,
- (7) F on L iff for A st $A \in F$ holds A on L ,
- (8) F on P iff for A st $A \in F$ holds A on P ,
- (9) F is_linear iff ex L st F on L ,
- (10) F is_planar iff ex P st F on P ,
- (11) $\{A, B\}$ on L iff A on L & B on L ,
- (12) $\{A, B, C\}$ on L iff A on L & B on L & C on L ,
- (13) $\{A, B\}$ on P iff A on P & B on P ,
- (14) $\{A, B, C\}$ on P iff A on P & B on P & C on P ,
- (15) $\{A, B, C, D\}$ on P iff A on P & B on P & C on P & D on P ,
- (16) $G \subseteq F$ & F on L implies G on L ,
- (17) $G \subseteq F$ & F on P implies G on P ,
- (18) F on L & A on L iff $F \cup \{A\}$ on L ,
- (19) F on P & A on P iff $F \cup \{A\}$ on P ,
- (20) $F \cup G$ on L iff F on L & G on L ,
- (21) $F \cup G$ on P iff F on P & G on P ,
- (22) $G \subseteq F$ & F is_linear implies G is_linear ,
- (23) $G \subseteq F$ & F is_planar implies G is_planar .

The mode

IncSpace,

which widens to the type IncStruct, is defined by

(for L being LINE of it ex A, B being POINT of it st $A \neq B$ & $\{A, B\}$ on L) &
 (for A, B being POINT of it ex L being LINE of it st $\{A, B\}$ on L) &
 (for A, B being POINT of it, K, L being LINE of it
 st $A \neq B$ & $\{A, B\}$ on K & $\{A, B\}$ on L holds $K = L$)
 & (for P being PLANE of it ex A being POINT of it st A on P) &

(for A, B, C being POINT of it ex P being PLANE of it st $\{A, B, C\}$ on P) &
(for A, B, C being POINT of it, P, Q being PLANE of it
st not $\{A, B, C\}$ is_linear & $\{A, B, C\}$ on P & $\{A, B, C\}$ on Q holds $P = Q$)
&
(for L being LINE of it, P being PLANE of it
st ex A, B being POINT of it st $A \neq B$ & $\{A, B\}$ on L & $\{A, B\}$ on P holds L on P)
&
(for A being POINT of it, P, Q being PLANE of it
st A on P & A on Q ex B being POINT of it st $A \neq B$ & B on P & B on Q)
& (ex A, B, C, D being POINT of it st not $\{A, B, C, D\}$ is_planar) &
for A being POINT of it, L being LINE of it, P being PLANE of it
st A on L & L on P holds A on P .

The following proposition is true

(24) (for L being LINE of S ex A, B being POINT of S st $A \neq B$ & $\{A, B\}$ on L)
& (for A, B being POINT of S ex L being LINE of S st $\{A, B\}$ on L) &
(for A, B being POINT of S , K, L being LINE of S
st $A \neq B$ & $\{A, B\}$ on K & $\{A, B\}$ on L holds $K = L$)
& (for P being PLANE of S ex A being POINT of S st A on P) &
(for A, B, C being POINT of S ex P being PLANE of S st $\{A, B, C\}$ on P)
&
(for A, B, C being POINT of S , P, Q being PLANE of S
st not $\{A, B, C\}$ is_linear & $\{A, B, C\}$ on P & $\{A, B, C\}$ on Q holds $P = Q$)
&
(for L being LINE of S , P being PLANE of S st
ex A, B being POINT of S st $A \neq B$ & $\{A, B\}$ on L & $\{A, B\}$ on P
holds L on P)
&
(for A being POINT of S , P, Q being PLANE of S
st A on P & A on Q ex B being POINT of S st $A \neq B$ & B on P & B on Q)
& (ex A, B, C, D being POINT of S st not $\{A, B, C, D\}$ is_planar) & (
for A being POINT of S , L being LINE of S , P being PLANE of S
st A on L & L on P holds A on P)
implies S is IncSpace.

For simplicity we adopt the following convention: S will denote an object of the type IncSpace; A, B, C, D will denote objects of the type POINT of S ; $K, L, L1, L2$ will denote objects of the type LINE of S ; P, Q will denote objects of the type PLANE of S ; F will denote an object of the type Subset of the Points of S . The following propositions are true:

- (25) $\text{ex } A, B \text{ st } A \neq B \ \& \ \{A, B\} \text{ on } L,$
- (26) $\text{ex } L \text{ st } \{A, B\} \text{ on } L,$
- (27) $A \neq B \ \& \ \{A, B\} \text{ on } K \ \& \ \{A, B\} \text{ on } L \text{ implies } K = L,$
- (28) $\text{ex } A \text{ st } A \text{ on } P,$
- (29) $\text{ex } P \text{ st } \{A, B, C\} \text{ on } P,$
- (30) $\text{not } \{A, B, C\} \text{ is_linear} \ \& \ \{A, B, C\} \text{ on } P \ \& \ \{A, B, C\} \text{ on } Q \text{ implies } P = Q,$
- (31) $(\text{ex } A, B \text{ st } A \neq B \ \& \ \{A, B\} \text{ on } L \ \& \ \{A, B\} \text{ on } P) \text{ implies } L \text{ on } P,$
- (32) $A \text{ on } P \ \& \ A \text{ on } Q \text{ implies } \text{ex } B \text{ st } A \neq B \ \& \ B \text{ on } P \ \& \ B \text{ on } Q,$
- (33) $\text{ex } A, B, C, D \text{ st } \text{not } \{A, B, C, D\} \text{ is_planar},$
- (34) $A \text{ on } L \ \& \ L \text{ on } P \text{ implies } A \text{ on } P,$
- (35) $F \text{ on } L \ \& \ L \text{ on } P \text{ implies } F \text{ on } P,$
- (36) $\{A, A, B\} \text{ is_linear},$
- (37) $\{A, A, B, C\} \text{ is_planar},$
- (38) $\{A, B, C\} \text{ is_linear} \text{ implies } \{A, B, C, D\} \text{ is_planar},$
- (39) $A \neq B \ \& \ \{A, B\} \text{ on } L \ \& \ \text{not } C \text{ on } L \text{ implies } \text{not } \{A, B, C\} \text{ is_linear},$
- (40) $\text{not } \{A, B, C\} \text{ is_linear} \ \& \ \{A, B, C\} \text{ on } P \ \& \ \text{not } D \text{ on } P$
 $\text{implies } \text{not } \{A, B, C, D\} \text{ is_planar},$
- (41) $\text{not } (\text{ex } P \text{ st } K \text{ on } P \ \& \ L \text{ on } P) \text{ implies } K \neq L,$
- (42) $\text{not } (\text{ex } P \text{ st } L \text{ on } P \ \& \ L1 \text{ on } P \ \& \ L2 \text{ on } P)$
 $\ \& \ (\text{ex } A \text{ st } A \text{ on } L \ \& \ A \text{ on } L1 \ \& \ A \text{ on } L2)$
 $\text{implies } L \neq L1,$
- (43) $L1 \text{ on } P \ \& \ L2 \text{ on } P \ \& \ \text{not } L \text{ on } P \ \& \ L1 \neq L2$
 $\text{implies } \text{not } \text{ex } Q \text{ st } L \text{ on } Q \ \& \ L1 \text{ on } Q \ \& \ L2 \text{ on } Q,$

- (44) $\text{ex } P \text{ st } A \text{ on } P \ \& \ L \text{ on } P,$
- (45) $(\text{ex } A \text{ st } A \text{ on } K \ \& \ A \text{ on } L) \text{ implies ex } P \text{ st } K \text{ on } P \ \& \ L \text{ on } P,$
- (46) $A \neq B \text{ implies ex } L \text{ st for } K \text{ holds } \{A, B\} \text{ on } K \text{ iff } K = L,$
- (47) $\text{not } \{A, B, C\} \text{ is_linear}$
 $\text{implies ex } P \text{ st for } Q \text{ holds } \{A, B, C\} \text{ on } Q \text{ iff } P = Q,$
- (48) $\text{not } A \text{ on } L \text{ implies ex } P \text{ st for } Q \text{ holds } A \text{ on } Q \ \& \ L \text{ on } Q \text{ iff } P = Q,$
- (49) $K \neq L \ \& \ (\text{ex } A \text{ st } A \text{ on } K \ \& \ A \text{ on } L)$
 $\text{implies ex } P \text{ st for } Q \text{ holds } K \text{ on } Q \ \& \ L \text{ on } Q \text{ iff } P = Q.$

Let us consider S, A, B . Assume that the following holds

$$A \neq B.$$

The functor

$$\text{Line}(A, B),$$

with values of the type LINE of S , is defined by

$$\{A, B\} \text{ on it}.$$

Let us consider S, A, B, C . Assume that the following holds

$$\text{not } \{A, B, C\} \text{ is_linear}.$$

The functor

$$\text{Plane}(A, B, C),$$

yields the type PLANE of S and is defined by

$$\{A, B, C\} \text{ on it}.$$

Let us consider S, A, L . Assume that the following holds

$$\text{not } A \text{ on } L.$$

The functor

$$\text{Plane}(A, L),$$

with values of the type PLANE of S , is defined by

$$A \text{ on it} \ \& \ L \text{ on it}.$$

Let us consider S, K, L . Assume that the following holds

$$K \neq L.$$

Moreover we assume that

$$\text{ex } A \text{ st } A \text{ on } K \ \& \ A \text{ on } L.$$

The functor

$$\text{Plane}(K, L),$$

with values of the type PLANE of S , is defined by

$$K \text{ on it } \& \ L \text{ on it}.$$

Next we state a number of propositions:

- (50) $A \neq B$ **implies** $\{A, B\}$ on Line (A, B) ,
- (51) $A \neq B$ & $\{A, B\}$ on K **implies** $K = \text{Line}(A, B)$,
- (52) **not** $\{A, B, C\}$ is_linear **implies** $\{A, B, C\}$ on Plane (A, B, C) ,
- (53) **not** $\{A, B, C\}$ is_linear & $\{A, B, C\}$ on Q **implies** $Q = \text{Plane}(A, B, C)$,
- (54) **not** A on L **implies** A on Plane (A, L) & L on Plane (A, L) ,
- (55) **not** A on L & A on Q & L on Q **implies** $Q = \text{Plane}(A, L)$,
- (56) $K \neq L$ & (**ex** A **st** A on K & A on L)
implies K on Plane (K, L) & L on Plane (K, L) ,
- (57) $A \neq B$ **implies** Line $(A, B) = \text{Line}(B, A)$,
- (58) **not** $\{A, B, C\}$ is_linear **implies** Plane $(A, B, C) = \text{Plane}(A, C, B)$,
- (59) **not** $\{A, B, C\}$ is_linear **implies** Plane $(A, B, C) = \text{Plane}(B, A, C)$,
- (60) **not** $\{A, B, C\}$ is_linear **implies** Plane $(A, B, C) = \text{Plane}(B, C, A)$,
- (61) **not** $\{A, B, C\}$ is_linear **implies** Plane $(A, B, C) = \text{Plane}(C, A, B)$,
- (62) **not** $\{A, B, C\}$ is_linear **implies** Plane $(A, B, C) = \text{Plane}(C, B, A)$,
- (63) $K \neq L$ & (**ex** A **st** A on K & A on L) & K on Q & L on Q
implies $Q = \text{Plane}(K, L)$,
- (64) $K \neq L$ & (**ex** A **st** A on K & A on L) **implies** Plane $(K, L) = \text{Plane}(L, K)$,
- (65) $A \neq B$ & C on Line (A, B) **implies** $\{A, B, C\}$ is_linear ,
- (66) $A \neq B$ & $A \neq C$ & $\{A, B, C\}$ is_linear **implies** Line $(A, B) = \text{Line}(A, C)$,
- (67) **not** $\{A, B, C\}$ is_linear **implies** Plane $(A, B, C) = \text{Plane}(C, \text{Line}(A, B))$,

- (68) **not** $\{A, B, C\}$ is_linear & D on Plane (A, B, C)
implies $\{A, B, C, D\}$ is_planar ,
- (69) **not** C on L & $\{A, B\}$ on L & $A \neq B$ **implies** Plane $(C, L) =$ Plane (A, B, C) ,
- (70) **not** $\{A, B, C\}$ is_linear
implies Plane $(A, B, C) =$ Plane $($ Line $(A, B),$ Line $(A, C))$,
- (71) **ex** A, B, C **st** $\{A, B, C\}$ on P & **not** $\{A, B, C\}$ is_linear ,
- (72) **ex** A, B, C, D **st** A on P & **not** $\{A, B, C, D\}$ is_planar ,
- (73) **ex** B **st** $A \neq B$ & B on L ,
- (74) $A \neq B$ **implies** **ex** C **st** C on P & **not** $\{A, B, C\}$ is_linear ,
- (75) **not** $\{A, B, C\}$ is_linear **implies** **ex** D **st** **not** $\{A, B, C, D\}$ is_planar ,
- (76) **ex** B, C **st** $\{B, C\}$ on P & **not** $\{A, B, C\}$ is_linear ,
- (77) $A \neq B$ **implies** **ex** C, D **st** **not** $\{A, B, C, D\}$ is_planar ,
- (78) **ex** B, C, D **st** **not** $\{A, B, C, D\}$ is_planar ,
- (79) **ex** L **st** **not** A on L & L on P ,
- (80) A on P **implies** **ex** $L, L1, L2$ **st** $L1 \neq L2$
& $L1$ on P & $L2$ on P & **not** L on P & A on L & A on $L1$ & A on $L2$,
- (81) **ex** $L, L1, L2$
st A on L & A on $L1$ & A on $L2$ & **not** **ex** P **st** L on P & $L1$ on P & $L2$ on P ,
- (82) **ex** P **st** A on P & **not** L on P ,
- (83) **ex** A **st** A on P & **not** A on L ,
- (84) **ex** K **st** **not** **ex** P **st** L on P & K on P ,
- (85) **ex** P, Q **st** $P \neq Q$ & L on P & L on Q ,
- (86) $K \neq L$ & $\{A, B\}$ on K & $\{A, B\}$ on L **implies** $A = B$,
- (87) **not** L on P & $\{A, B\}$ on L & $\{A, B\}$ on P **implies** $A = B$,
- (88) $P \neq Q$ **implies** **not** (**ex** A **st** A on P & A on Q)
or **ex** L **st** **for** B **holds** B on P & B on Q **iff** B on L .

References

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