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A Model of ZF Set Theory Language

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Summary. The goal of this article is to construct a language of the ZF set theory and to develop a notational and conceptual base which facilitates a convenient usage of the language.

The articles [5], [6], [3], [4], [1], and [2] provide the terminology and notation for this paper. For simplicity we adopt the following convention: k, n will have the type Nat; D will have the type DOMAIN; a will have the type Any; p, q will have the type FinSequence **of** NAT. The constant VAR has the type SUBDOMAIN **of** NAT, and is defined by

$$\mathbf{it} = \{ k : 5 \leq k \}.$$

The following proposition is true

(1) $$\text{VAR} = \{ k : 5 \leq k \}.$$

Variable stands for Element **of** VAR.

One can prove the following proposition

(2) a **is** Variable **iff** a **is** Element **of** VAR.

Let us consider n. The functor

$$\xi\, n,$$

with values of the type Variable, is defined by

$$\mathbf{it} = 5 + n.$$

One can prove the following proposition

(3) $$\xi\, n = 5 + n.$$

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In the sequel x, y, z, t denote objects of the type Variable. Let us consider x. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$<x> \quad \text{is} \quad \text{FinSequence } \textbf{of} \text{ NAT}.$$

We now define two new functors. Let us consider x, y. The functor

$$x \mathrel{=} y,$$

with values of the type FinSequence **of** NAT, is defined by

$$\textbf{it} = <0> ^\frown <x> ^\frown <y>.$$

The functor

$$x \mathrel{\epsilon} y,$$

yields the type FinSequence **of** NAT and is defined by

$$\textbf{it} = <1> ^\frown <x> ^\frown <y>.$$

Next we state four propositions:

(4) $$x \mathrel{=} y = <0> ^\frown <x> ^\frown <y>,$$

(5) $$x \mathrel{\epsilon} y = <1> ^\frown <x> ^\frown <y>,$$

(6) $$x \mathrel{=} y = z \mathrel{=} t \ \textbf{implies}\ x = z \ \&\ y = t,$$

(7) $$x \mathrel{\epsilon} y = z \mathrel{\epsilon} t \ \textbf{implies}\ x = z \ \&\ y = t.$$

We now define two new functors. Let us consider p. The functor

$$\neg p,$$

with values of the type FinSequence **of** NAT, is defined by

$$\textbf{it} = <2> ^\frown p.$$

Let us consider q. The functor

$$p \wedge q,$$

with values of the type FinSequence **of** NAT, is defined by

$$\textbf{it} = <3> ^\frown p ^\frown q.$$

Next we state three propositions:

(8) $$\neg p = <2> ^\frown p,$$

(9) $$p \wedge q = <3> ^\frown p ^\frown q,$$

(10) $$\neg p = \neg q \ \textbf{implies}\ p = q.$$

Let us consider x, p. The functor

$$\forall\,(x,p),$$

yields the type FinSequence **of** NAT and is defined by

$$\textbf{it} = <4> \frown <x> \frown p.$$

The following propositions are true:

(11) $$\forall\,(x,p) = <4> \frown <x> \frown p,$$

(12) $$\forall\,(x,p) = \forall\,(y,q) \textbf{ implies } x = y \;\&\; p = q.$$

The constant WFF has the type DOMAIN, and is defined by

$$(\textbf{for } a \textbf{ st } a \in \textbf{it} \textbf{ holds } a \textbf{ is } \text{FinSequence } \textbf{of } \text{NAT}) \;\&$$
$$(\textbf{for } x,y \textbf{ holds } x = y \in \textbf{it} \;\&\; x \epsilon y \in \textbf{it}) \;\&\; (\textbf{for } p \textbf{ st } p \in \textbf{it} \textbf{ holds } \neg p \in \textbf{it}) \;\&$$
$$(\textbf{for } p,q \textbf{ st } p \in \textbf{it} \;\&\; q \in \textbf{it} \textbf{ holds } p \wedge q \in \textbf{it}) \;\&\; (\textbf{for } x,p \textbf{ st } p \in \textbf{it} \textbf{ holds } \forall\,(x,p) \in \textbf{it}) \;\&$$
$$\textbf{for } D \textbf{ st}$$
$$(\textbf{for } a \textbf{ st } a \in D \textbf{ holds } a \textbf{ is } \text{FinSequence } \textbf{of } \text{NAT}) \;\&$$
$$(\textbf{for } x,y \textbf{ holds } x = y \in D \;\&\; x \epsilon y \in D) \;\&\; (\textbf{for } p \textbf{ st } p \in D \textbf{ holds } \neg p \in D)$$
$$\&\; (\textbf{for } p,q \textbf{ st } p \in D \;\&\; q \in D \textbf{ holds } p \wedge q \in D) \;\&\; \textbf{for } x,p \textbf{ st } p \in D \textbf{ holds } \forall\,(x,p) \in D$$
$$\textbf{holds it} \subseteq D.$$

One can prove the following proposition

(13) $$(\textbf{for } a \textbf{ st } a \in \text{WFF} \textbf{ holds } a \textbf{ is } \text{FinSequence } \textbf{of } \text{NAT}) \;\&$$
$$(\textbf{for } x,y \textbf{ holds } x = y \in \text{WFF} \;\&\; x \epsilon y \in \text{WFF}) \;\&$$
$$(\textbf{for } p \textbf{ st } p \in \text{WFF} \textbf{ holds } \neg p \in \text{WFF}) \;\&$$
$$(\textbf{for } p,q \textbf{ st } p \in \text{WFF} \;\&\; q \in \text{WFF} \textbf{ holds } p \wedge q \in \text{WFF}) \;\&$$
$$(\textbf{for } x,p \textbf{ st } p \in \text{WFF} \textbf{ holds } \forall\,(x,p) \in \text{WFF}) \;\&\; \textbf{for } D \textbf{ st}$$
$$(\textbf{for } a \textbf{ st } a \in D \textbf{ holds } a \textbf{ is } \text{FinSequence } \textbf{of } \text{NAT}) \;\&$$
$$(\textbf{for } x,y \textbf{ holds } x = y \in D \;\&\; x \epsilon y \in D) \;\&\; (\textbf{for } p \textbf{ st } p \in D \textbf{ holds } \neg p \in D) \;\&$$
$$(\textbf{for } p,q \textbf{ st } p \in D \;\&\; q \in D \textbf{ holds } p \wedge q \in D)$$
$$\&\; \textbf{for } x,p \textbf{ st } p \in D \textbf{ holds } \forall\,(x,p) \in D$$
$$\textbf{holds } \text{WFF} \subseteq D.$$

The mode

ZF-formula,

which widens to the type FinSequence **of** NAT, is defined by

it is Element **of** WFF .

We now state two propositions:

(14) $$a \textbf{ is } \text{ZF-formula} \textbf{ iff } a \in \text{WFF}\,,$$

(15) $$a \textbf{ is } \text{ZF-formula} \textbf{ iff } a \textbf{ is } \text{Element} \textbf{ of } \text{WFF}\,.$$

In the sequel F, $F1$, G, $G1$, H, $H1$ denote objects of the type ZF-formula. Let us consider x, y. Let us note that it makes sense to consider the following functors on restricted areas. Then

$$x = y \qquad \text{is} \qquad \text{ZF-formula}\,,$$
$$x \epsilon y \qquad \text{is} \qquad \text{ZF-formula}\,.$$

Let us consider H. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\neg H \qquad \text{is} \qquad \text{ZF-formula}\,.$$

Let us consider G. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$H \wedge G \qquad \text{is} \qquad \text{ZF-formula}\,.$$

Let us consider x, H. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\forall\,(x, H) \qquad \text{is} \qquad \text{ZF-formula}\,.$$

We now define five new predicates. Let us consider H. The predicate

$$H \text{ is\_a\_equality} \qquad \text{is defined by} \qquad \textbf{ex}\, x{,}y \textbf{ st } H = x = y.$$

The predicate

$$H \text{ is\_a\_membership} \qquad \text{is defined by} \qquad \textbf{ex}\, x{,}y \textbf{ st } H = x \epsilon y.$$

The predicate

$$H \text{ is\_negative} \qquad \text{is defined by} \qquad \textbf{ex}\, H1 \textbf{ st } H = \neg H1.$$

The predicate

$$H \text{ is\_conjunctive} \qquad \text{is defined by} \qquad \textbf{ex}\, F{,}G \textbf{ st } H = F \wedge G.$$

The predicate

$$H \text{ is\_universal} \qquad \text{is defined by} \qquad \textbf{ex}\, x{,}H1 \textbf{ st } H = \forall\,(x, H1).$$

The following proposition is true

(16) $$\begin{gathered}(H \text{ is\_a\_equality} \textbf{ iff } \textbf{ex}\, x{,}y \textbf{ st } H = x = y)\ \&\\ (H \text{ is\_a\_membership} \textbf{ iff } \textbf{ex}\, x{,}y \textbf{ st } H = x \epsilon y)\ \&\\ (H \text{ is\_negative} \textbf{ iff } \textbf{ex}\, H1 \textbf{ st } H = \neg H1)\ \&\\ (H \text{ is\_conjunctive} \textbf{ iff } \textbf{ex}\, F{,}G \textbf{ st } H = F \wedge G)\\ \&\ (H \text{ is\_universal} \textbf{ iff } \textbf{ex}\, x{,}H1 \textbf{ st } H = \forall\,(x, H1)).\end{gathered}$$

Let us consider H. The predicate

$$H \text{ is\_atomic} \qquad \text{is defined by} \qquad H \text{ is\_a\_equality } \textbf{or } H \text{ is\_a\_membership}\,.$$

Next we state a proposition

(17) $$H \text{ is\_atomic } \textbf{iff } H \text{ is\_a\_equality } \textbf{or } H \text{ is\_a\_membership}\,.$$

We now define two new functors. Let us consider F, G. The functor

$$F \vee G,$$

yields the type ZF-formula and is defined by

$$\textbf{it} = \neg\,(\neg\,F \wedge \neg\,G).$$

The functor

$$F \Rightarrow G,$$

yields the type ZF-formula and is defined by

$$\textbf{it} = \neg\,(F \wedge \neg\,G).$$

The following two propositions are true:

(18) $$F \vee G = \neg\,(\neg\,F \wedge \neg\,G),$$

(19) $$F \Rightarrow G = \neg\,(F \wedge \neg\,G).$$

Let us consider F, G. The functor

$$F \Leftrightarrow G,$$

yields the type ZF-formula and is defined by

$$\textbf{it} = (F \Rightarrow G) \wedge (G \Rightarrow F).$$

We now state a proposition

(20) $$F \Leftrightarrow G = (F \Rightarrow G) \wedge (G \Rightarrow F).$$

Let us consider x, H. The functor

$$\exists\,(x, H),$$

yields the type ZF-formula and is defined by

$$\textbf{it} = \neg\,\forall\,(x, \neg\,H).$$

The following proposition is true

(21) $$\exists\,(x, H) = \neg\,\forall\,(x, \neg\,H).$$

We now define four new predicates. Let us consider H. The predicate

$$H \text{ is\_disjunctive} \qquad \text{is defined by} \qquad \textbf{ex}\, F,G \textbf{ st } H = F \vee G.$$

The predicate

$$H \text{ is\_conditional} \qquad \text{is defined by} \qquad \textbf{ex}\, F,G \textbf{ st } H = F \Rightarrow G.$$

The predicate

$$H \text{ is\_biconditional} \qquad \text{is defined by} \qquad \textbf{ex}\, F,G \textbf{ st } H = F \Leftrightarrow G.$$

The predicate

$$H \text{ is\_existential} \qquad \text{is defined by} \qquad \textbf{ex}\, x,H1 \textbf{ st } H = \exists\,(x, H1).$$

The following proposition is true

(22)
$$\begin{gathered}(H \text{ is\_disjunctive } \textbf{iff ex}\, F,G \textbf{ st } H = F \vee G)\ \&\\ (H \text{ is\_conditional } \textbf{iff ex}\, F,G \textbf{ st } H = F \Rightarrow G)\ \&\\ (H \text{ is\_biconditional } \textbf{iff ex}\, F,G \textbf{ st } H = F \Leftrightarrow G)\\ \&\ (H \text{ is\_existential } \textbf{iff ex}\, x,H1 \textbf{ st } H = \exists\,(x, H1)).\end{gathered}$$

We now define two new functors. Let us consider x, y, H. The functor

$$\forall\,(x, y, H),$$

yields the type ZF-formula and is defined by

$$\textbf{it} = \forall\,(x, \forall\,(y, H)).$$

The functor

$$\exists\,(x, y, H),$$

yields the type ZF-formula and is defined by

$$\textbf{it} = \exists\,(x, \exists\,(y, H)).$$

The following proposition is true

(23)
$$\forall\,(x, y, H) = \forall\,(x, \forall\,(y, H))\ \&\ \exists\,(x, y, H) = \exists\,(x, \exists\,(y, H)).$$

We now define two new functors. Let us consider x, y, z, H. The functor

$$\forall\,(x, y, z, H),$$

with values of the type ZF-formula, is defined by

$$\textbf{it} = \forall\,(x, \forall\,(y, z, H)).$$

The functor

$$\exists\,(x, y, z, H),$$

with values of the type ZF-formula, is defined by

$$\textbf{it} = \exists\,(x, \exists\,(y, z, H)).$$

We now state several propositions:

(24) $$\forall\,(x, y, z, H) = \forall\,(x, \forall\,(y, z, H)) \mathbin{\&} \exists\,(x, y, z, H) = \exists\,(x, \exists\,(y, z, H)),$$

(25) $$H \text{ is\_a\_equality} \;\textbf{or}\; H \text{ is\_a\_membership} \;\textbf{or}\; H \text{ is\_negative} \;\textbf{or}\; H \text{ is\_conjunctive} \;\textbf{or}\; H \text{ is\_universal}\,,$$

(26) $$H \text{ is\_atomic} \;\textbf{or}\; H \text{ is\_negative} \;\textbf{or}\; H \text{ is\_conjunctive} \;\textbf{or}\; H \text{ is\_universal}\,,$$

(27) $$H \text{ is\_atomic} \;\textbf{implies}\; \text{len}\, H = 3,$$

(28) $$H \text{ is\_atomic} \;\textbf{or ex}\; H1 \;\textbf{st}\; \text{len}\, H1 + 1 \leq \text{len}\, H,$$

(29) $$3 \leq \text{len}\, H,$$

(30) $$\text{len}\, H = 3 \;\textbf{implies}\; H \text{ is\_atomic}\,.$$

One can prove the following propositions:

(31) $$\textbf{for}\; x, y \;\textbf{holds}\; (x = y).1 = 0 \mathbin{\&} (x \epsilon y).1 = 1,$$

(32) $$\textbf{for}\; H \;\textbf{holds}\; (\neg H).1 = 2,$$

(33) $$\textbf{for}\; F, G \;\textbf{holds}\; (F \wedge G).1 = 3,$$

(34) $$\textbf{for}\; x, H \;\textbf{holds}\; \forall\,(x, H).1 = 4,$$

(35) $$H \text{ is\_a\_equality} \;\textbf{implies}\; H.1 = 0,$$

(36) $$H \text{ is\_a\_membership} \;\textbf{implies}\; H.1 = 1,$$

(37) $$H \text{ is\_negative} \;\textbf{implies}\; H.1 = 2,$$

(38) $$H \text{ is\_conjunctive} \;\textbf{implies}\; H.1 = 3,$$

(39) $$H \text{ is\_universal} \;\textbf{implies}\; H.1 = 4,$$

(40) $$H \text{ is\_a\_equality} \mathbin{\&} H.1 = 0 \;\textbf{or}\; H \text{ is\_a\_membership} \mathbin{\&} H.1 = 1 \;\textbf{or}\; H \text{ is\_negative} \mathbin{\&} H.1 = 2 \;\textbf{or}\; H \text{ is\_conjunctive} \mathbin{\&} H.1 = 3 \;\textbf{or}\; H \text{ is\_universal} \mathbin{\&} H.1 = 4,$$

(41) $$H.1 = 0 \;\textbf{implies}\; H \text{ is\_a\_equality}\,,$$

(42) $$H.1 = 1 \;\textbf{implies}\; H \text{ is\_a\_membership}\,,$$

(43) $$H.1 = 2 \textbf{ implies } H \text{ is\_negative},$$

(44) $$H.1 = 3 \textbf{ implies } H \text{ is\_conjunctive},$$

(45) $$H.1 = 4 \textbf{ implies } H \text{ is\_universal}.$$

In the sequel sq denotes an object of the type FinSequence. We now state several propositions:

(46) $$H = F ^\frown sq \textbf{ implies } H = F,$$

(47) $$H \wedge G = H1 \wedge G1 \textbf{ implies } H = H1 \ \&\ G = G1,$$

(48) $$F \vee G = F1 \vee G1 \textbf{ implies } F = F1 \ \&\ G = G1,$$

(49) $$F \Rightarrow G = F1 \Rightarrow G1 \textbf{ implies } F = F1 \ \&\ G = G1,$$

(50) $$F \Leftrightarrow G = F1 \Leftrightarrow G1 \textbf{ implies } F = F1 \ \&\ G = G1,$$

(51) $$\exists (x, H) = \exists (y, G) \textbf{ implies } x = y \ \&\ H = G.$$

We now define two new functors. Let us consider H. Assume that the following holds

$$H \text{ is\_atomic}.$$

The functor

$$\text{Var}_1 H,$$

yields the type Variable and is defined by

$$\textbf{it} = H.2.$$

The functor

$$\text{Var}_2 H,$$

yields the type Variable and is defined by

$$\textbf{it} = H.3.$$

One can prove the following three propositions:

(52) $$H \text{ is\_atomic } \textbf{implies } \text{Var}_1 H = H.2 \ \&\ \text{Var}_2 H = H.3,$$

(53) $$H \text{ is\_a\_equality } \textbf{implies } H = (\text{Var}_1 H) = \text{Var}_2 H,$$

(54) $$H \text{ is\_a\_membership } \textbf{implies } H = (\text{Var}_1 H) \epsilon \text{ Var}_2 H.$$

Let us consider H. Assume that the following holds

$$H \text{ is\_negative}.$$

The functor

$$\text{the\_argument\_of}\,H,$$

with values of the type ZF-formula, is defined by

$$\neg\,\mathbf{it} = H.$$

We now state a proposition

(55) H is_negative **implies** $H = \neg\,$the_argument_of H.

We now define two new functors. Let us consider H. Assume that the following holds

$$H \text{ is\_conjunctive } \mathbf{or}\ H \text{ is\_disjunctive}\,.$$

The functor

$$\text{the\_left\_argument\_of}\,H,$$

with values of the type ZF-formula, is defined by

$\mathbf{ex}\,H1$ **st** $\mathbf{it} \wedge H1 = H$, **if** H is_conjunctive ,
$\mathbf{ex}\,H1$ **st** $\mathbf{it} \vee H1 = H$, **otherwise**.

The functor

$$\text{the\_right\_argument\_of}\,H,$$

with values of the type ZF-formula, is defined by

$\mathbf{ex}\,H1$ **st** $H1 \wedge \mathbf{it} = H$, **if** H is_conjunctive ,
$\mathbf{ex}\,H1$ **st** $H1 \vee \mathbf{it} = H$, **otherwise**.

One can prove the following propositions:

(56) H is_conjunctive **implies** ($F =$ the_left_argument_of H **iff** **ex** G **st** $F \wedge G = H$) & ($F =$ the_right_argument_of H **iff** **ex** G **st** $G \wedge F = H$),

(57) H is_disjunctive **implies** ($F =$ the_left_argument_of H **iff** **ex** G **st** $F \vee G = H$) & ($F =$ the_right_argument_of H **iff** **ex** G **st** $G \vee F = H$),

(58) H is_conjunctive **implies** $H =$ (the_left_argument_of H) $\wedge$ the_right_argument_of H,

(59) H is_disjunctive **implies** $H =$ (the_left_argument_of H) $\vee$ the_right_argument_of H.

We now define two new functors. Let us consider H. Assume that the following holds

$$H \text{ is\_universal } \mathbf{or}\ H \text{ is\_existential}\,.$$

The functor

$$\text{bound\_in}\,H,$$

with values of the type Variable, is defined by

$$\begin{cases} \mathbf{ex}\,H1\ \mathbf{st}\ \forall\,(\mathbf{it}\,,H1) = H, & \mathbf{if}\quad H\ \text{is\_universal}\,, \\ \mathbf{ex}\,H1\ \mathbf{st}\ \exists\,(\mathbf{it}\,,H1) = H, & \mathbf{otherwise}. \end{cases}$$

The functor

$$\text{the\_scope\_of}\,H,$$

with values of the type ZF-formula, is defined by

$$\begin{cases} \mathbf{ex}\,x\ \mathbf{st}\ \forall\,(x, \mathbf{it}) = H, & \mathbf{if}\quad H\ \text{is\_universal}\,, \\ \mathbf{ex}\,x\ \mathbf{st}\ \exists\,(x, \mathbf{it}) = H, & \mathbf{otherwise}. \end{cases}$$

Next we state four propositions:

(60) $$H\ \text{is\_universal}\ \mathbf{implies}\ (x = \text{bound\_in}\,H\ \mathbf{iff}\ \mathbf{ex}\,H1\ \mathbf{st}\ \forall\,(x, H1) = H)$$
$$\&\ (H1 = \text{the\_scope\_of}\,H\ \mathbf{iff}\ \mathbf{ex}\,x\ \mathbf{st}\ \forall\,(x, H1) = H),$$

(61) $$H\ \text{is\_existential}\ \mathbf{implies}\ (x = \text{bound\_in}\,H\ \mathbf{iff}\ \mathbf{ex}\,H1\ \mathbf{st}\ \exists\,(x, H1) = H)$$
$$\&\ (H1 = \text{the\_scope\_of}\,H\ \mathbf{iff}\ \mathbf{ex}\,x\ \mathbf{st}\ \exists\,(x, H1) = H),$$

(62) $$H\ \text{is\_universal}\ \mathbf{implies}\ H = \forall\,(\text{bound\_in}\,H, \text{the\_scope\_of}\,H),$$

(63) $$H\ \text{is\_existential}\ \mathbf{implies}\ H = \exists\,(\text{bound\_in}\,H, \text{the\_scope\_of}\,H).$$

We now define two new functors. Let us consider H. Assume that the following holds

$$H\ \text{is\_conditional}\,.$$

The functor

$$\text{the\_antecedent\_of}\,H,$$

with values of the type ZF-formula, is defined by

$$\mathbf{ex}\,H1\ \mathbf{st}\ H = \mathbf{it} \Rightarrow H1.$$

The functor

$$\text{the\_consequent\_of}\,H,$$

with values of the type ZF-formula, is defined by

$$\mathbf{ex}\,H1\ \mathbf{st}\ H = H1 \Rightarrow \mathbf{it}\,.$$

The following propositions are true:

(64) $$H\ \text{is\_conditional}\ \mathbf{implies}\ (F = \text{the\_antecedent\_of}\,H\ \mathbf{iff}\ \mathbf{ex}\,G\ \mathbf{st}\ H = F \Rightarrow G)$$
$$\&\ (F = \text{the\_consequent\_of}\,H\ \mathbf{iff}\ \mathbf{ex}\,G\ \mathbf{st}\ H = G \Rightarrow F),$$

(65) H is_conditional **implies** $H =$ (the_antecedent_of H) $\Rightarrow$ the_consequent_of H.

We now define two new functors. Let us consider H. Assume that the following holds

$$H \text{ is\_biconditional}.$$

The functor

$$\text{the\_left\_side\_of } H,$$

yields the type ZF-formula and is defined by

$$\textbf{ex } H1 \textbf{ st } H = \textbf{it} \Leftrightarrow H1.$$

The functor

$$\text{the\_right\_side\_of } H,$$

with values of the type ZF-formula, is defined by

$$\textbf{ex } H1 \textbf{ st } H = H1 \Leftrightarrow \textbf{it}.$$

We now state two propositions:

(66) H is_biconditional **implies** ($F =$ the_left_side_of H **iff ex** G **st** $H = F \Leftrightarrow G$) & ($F =$ the_right_side_of H **iff ex** G **st** $H = G \Leftrightarrow F$),

(67) H is_biconditional **implies** $H =$ (the_left_side_of H) $\Leftrightarrow$ the_right_side_of H.

Let us consider H, F. The predicate

$$H \text{ is\_immediate\_constituent\_of } F$$

is defined by

$$F = \neg H \textbf{ or } (\textbf{ex } H1 \textbf{ st } F = H \wedge H1 \textbf{ or } F = H1 \wedge H) \textbf{ or ex } x \textbf{ st } F = \forall (x, H).$$

We now state a number of propositions:

(68) H is_immediate_constituent_of F **iff** $F = \neg H$ **or** (**ex** $H1$ **st** $F = H \wedge H1$ **or** $F = H1 \wedge H$) **or ex** x **st** $F = \forall (x, H)$,

(69) **not** H is_immediate_constituent_of $x = y$,

(70) **not** H is_immediate_constituent_of $x \in y$,

(71) F is_immediate_constituent_of $\neg H$ **iff** $F = H$,

(72) F is_immediate_constituent_of $G \wedge H$ **iff** $F = G$ **or** $F = H$,

(73) F is_immediate_constituent_of $\forall (x, H)$ **iff** $F = H$,

(74) H is_atomic **implies not** F is_immediate_constituent_of H,

(75) H is_negative
implies (F is_immediate_constituent_of H **iff** F = the_argument_of H),

(76) H is_conjunctive **implies** (F is_immediate_constituent_of H
iff F = the_left_argument_of H **or** F = the_right_argument_of H),

(77) H is_universal
implies (F is_immediate_constituent_of H **iff** F = the_scope_of H).

In the sequel L will denote an object of the type FinSequence. Let us consider H, F. The predicate

H is_subformula_of F

is defined by

ex n,L **st** $1 \leq n$ & len $L = n$ & $L.1 = H$ & $L.n = F$ & **for** k **st** $1 \leq k$ & $k < n$
ex $H1$,$F1$ **st** $L.k = H1$ & $L.(k+1) = F1$ & $H1$ is_immediate_constituent_of $F1$.

Next we state two propositions:

(78) H is_subformula_of F **iff ex** n,L **st** $1 \leq n$ & len $L = n$ & $L.1 = H$ & $L.n = F$ &
for k **st** $1 \leq k$ & $k < n$ **ex** $H1$,$F1$
st $L.k = H1$ & $L.(k+1) = F1$ & $H1$ is_immediate_constituent_of $F1$,

(79) H is_subformula_of H.

Let us consider H, F. The predicate

H is_proper_subformula_of F is defined by H is_subformula_of F & $H \neq F$.

We now state several propositions:

(80) H is_proper_subformula_of F **iff** H is_subformula_of F & $H \neq F$,

(81) H is_immediate_constituent_of F **implies** len $H <$ len F,

(82) H is_immediate_constituent_of F **implies** H is_proper_subformula_of F,

(83) H is_proper_subformula_of F **implies** len $H <$ len F,

(84) H is_proper_subformula_of F
implies ex G **st** G is_immediate_constituent_of F.

The following propositions are true:

(85) F is_proper_subformula_of G & G is_proper_subformula_of H
implies F is_proper_subformula_of H,

$$F \text{ is\_subformula\_of } G \;\&\; G \text{ is\_subformula\_of } H \textbf{ implies } F \text{ is\_subformula\_of } H, \tag{86}$$

$$G \text{ is\_subformula\_of } H \;\&\; H \text{ is\_subformula\_of } G \textbf{ implies } G = H, \tag{87}$$

$$\textbf{not } F \text{ is\_proper\_subformula\_of } x = y, \tag{88}$$

$$\textbf{not } F \text{ is\_proper\_subformula\_of } x \,\epsilon\, y, \tag{89}$$

$$F \text{ is\_proper\_subformula\_of } \neg H \textbf{ implies } F \text{ is\_subformula\_of } H, \tag{90}$$

$$F \text{ is\_proper\_subformula\_of } G \wedge H \textbf{ implies } F \text{ is\_subformula\_of } G \textbf{ or } F \text{ is\_subformula\_of } H, \tag{91}$$

$$F \text{ is\_proper\_subformula\_of } \forall (x, H) \textbf{ implies } F \text{ is\_subformula\_of } H, \tag{92}$$

$$H \text{ is\_atomic } \textbf{implies not } F \text{ is\_proper\_subformula\_of } H, \tag{93}$$

$$H \text{ is\_negative } \textbf{implies } \text{the\_argument\_of } H \text{ is\_proper\_subformula\_of } H, \tag{94}$$

$$H \text{ is\_conjunctive } \textbf{implies } \text{the\_left\_argument\_of } H \text{ is\_proper\_subformula\_of } H \;\&\; \text{the\_right\_argument\_of } H \text{ is\_proper\_subformula\_of } H, \tag{95}$$

$$H \text{ is\_universal } \textbf{implies } \text{the\_scope\_of } H \text{ is\_proper\_subformula\_of } H, \tag{96}$$

$$H \text{ is\_subformula\_of } x = y \textbf{ iff } H = x = y, \tag{97}$$

$$H \text{ is\_subformula\_of } x \,\epsilon\, y \textbf{ iff } H = x \,\epsilon\, y. \tag{98}$$

Let us consider H. The functor

$$\text{Subformulae}\, H,$$

yields the type set and is defined by

$$a \in \textbf{it iff ex } F \textbf{ st } F = a \;\&\; F \text{ is\_subformula\_of } H.$$

We now state a number of propositions:

$$a \in \text{Subformulae}\, H \textbf{ iff ex } F \textbf{ st } F = a \;\&\; F \text{ is\_subformula\_of } H, \tag{99}$$

$$G \in \text{Subformulae}\, H \textbf{ implies } G \text{ is\_subformula\_of } H, \tag{100}$$

$$F \text{ is\_subformula\_of } H \textbf{ implies } \text{Subformulae}\, F \subseteq \text{Subformulae}\, H, \tag{101}$$

$$\text{Subformulae}\, x = y = \{x = y\}, \tag{102}$$

$$\text{Subformulae}\, x \,\epsilon\, y = \{x \,\epsilon\, y\}, \tag{103}$$

$$\text{Subformulae}\, \neg H = \text{Subformulae}\, H \cup \{\neg H\}, \tag{104}$$

$$\text{Subformulae}\,(H \wedge F) = \text{Subformulae}\,H \cup \text{Subformulae}\,F \cup \{H \wedge F\}, \tag{105}$$

$$\text{Subformulae}\,\forall\,(x, H) = \text{Subformulae}\,H \cup \{\forall\,(x, H)\}, \tag{106}$$

$$H\ \text{is\_atomic}\ \textbf{iff}\ \text{Subformulae}\,H = \{H\}, \tag{107}$$

$$H\ \text{is\_negative}\ \textbf{implies}\ \text{Subformulae}\,H = \text{Subformulae the\_argument\_of}\,H \cup \{H\}, \tag{108}$$

$$H\ \text{is\_conjunctive}\ \textbf{implies}\ \text{Subformulae}\,H = \text{Subformulae the\_left\_argument\_of}\,H \cup \text{Subformulae the\_right\_argument\_of}\,H \cup \{H\}, \tag{109}$$

$$H\ \text{is\_universal}\ \textbf{implies}\ \text{Subformulae}\,H = \text{Subformulae the\_scope\_of}\,H \cup \{H\}, \tag{110}$$

$$(H\ \text{is\_immediate\_constituent\_of}\ G\ \textbf{or}\ H\ \text{is\_proper\_subformula\_of}\ G\ \textbf{or}\ H\ \text{is\_subformula\_of}\ G)\ \&\ G \in \text{Subformulae}\,F\ \textbf{implies}\ H \in \text{Subformulae}\,F. \tag{111}$$

In the article we present several logical schemes. The scheme *ZF_Ind* deals with a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for}\ H\ \textbf{holds}\ \mathcal{P}[H]$$

provided the parameter satisfies the following conditions:

- $\textbf{for}\ H\ \textbf{st}\ H\ \text{is\_atomic}\ \textbf{holds}\ \mathcal{P}[H],$
- $\textbf{for}\ H\ \textbf{st}\ H\ \text{is\_negative}\ \&\ \mathcal{P}[\text{the\_argument\_of}\,H]\ \textbf{holds}\ \mathcal{P}[H],$
- $\textbf{for}\ H\ \textbf{st}\ H\ \text{is\_conjunctive}\ \&\ \mathcal{P}[\text{the\_left\_argument\_of}\,H]\ \&\ \mathcal{P}[\text{the\_right\_argument\_of}\,H]\ \textbf{holds}\ \mathcal{P}[H],$
- $\textbf{for}\ H\ \textbf{st}\ H\ \text{is\_universal}\ \&\ \mathcal{P}[\text{the\_scope\_of}\,H]\ \textbf{holds}\ \mathcal{P}[H].$

The scheme *ZF_CompInd* deals with a unary predicate $\mathcal{P}$ states that the following holds

$$\textbf{for}\ H\ \textbf{holds}\ \mathcal{P}[H]$$

provided the parameter satisfies the following condition:

- $\textbf{for}\ H\ \textbf{st}\ \textbf{for}\ F\ \textbf{st}\ F\ \text{is\_proper\_subformula\_of}\ H\ \textbf{holds}\ \mathcal{P}[F]\ \textbf{holds}\ \mathcal{P}[H].$

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