

The Fundamental Logic Structure in Quantum Mechanics

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Summary. In this article we present the logical structure given by four axioms of Mackey [3] in the set of propositions of Quantum Mechanics. The equivalence relation ($\text{PropRel}(Q)$) in the set of propositions ($\text{Prop } Q$) for given Quantum Mechanics Q is considered. The main text for this article is [6] where the structure of quotient space and the properties of equivalence relations, classes and partitions are studied.

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The articles [10], [1], [4], [2], [9], [8], [7], [5], and [6] provide the notation and terminology for this paper. In the sequel x will be arbitrary, X will be a non-empty set, and X_1 will be a set. Let us consider X , and let S be a σ -field of subsets of X . The functor probabilities S yields a non-empty set and is defined by:

$x \in \text{probabilities } S$ if and only if x is a probability on S .

We now state a proposition

- (1) For every σ -field S of subsets of X holds $x \in \text{probabilities } S$ if and only if x is a probability on S .

We consider quantum mechanics structures which are systems

$\langle \text{observables, states, a probability} \rangle$

where the observables, the states are non-empty sets and the probability is a function from $\{ \text{the observables, the states} \}$ into probabilities the Borel sets. In the sequel Q denotes a quantum mechanics structure. We now define two new functors. Let us consider Q . The functor $\text{Obs } Q$ yields a non-empty set and is defined by:

$\text{Obs } Q = \text{the observables of } Q$.

The functor $\text{Sts } Q$ yields a non-empty set and is defined by:

$\text{Sts } Q = \text{the states of } Q$.

The following propositions are true:

- (2) $\text{Obs } Q =$ the observables of Q .
- (3) $\text{Sts } Q =$ the states of Q .

We adopt the following convention: A_1, A_2 will denote elements of $\text{Obs } Q$, s, s_1, s_2 will denote elements of $\text{Sts } Q$, and E will denote an event of the Borel sets. Let us consider Q, A_1, s . The functor $\text{Meas}(A_1, s)$ yielding a probability on the Borel sets, is defined as follows:

$$\text{Meas}(A_1, s) = (\text{the probability of } Q)(\langle A_1, s \rangle).$$

One can prove the following proposition

- (4) $\text{Meas}(A_1, s) = (\text{the probability of } Q)(\langle A_1, s \rangle)$.

A quantum mechanics structure is said to be a quantum mechanics if:

- (i) for all elements A_1, A_2 of $\text{Obs } Q$ such that for every element s of $\text{Sts } Q$ it holds $\text{Meas}(A_1, s) = \text{Meas}(A_2, s)$ holds $A_1 = A_2$,
- (ii) for all elements s_1, s_2 of $\text{Sts } Q$ such that for every element A of $\text{Obs } Q$ it holds $\text{Meas}(A, s_1) = \text{Meas}(A, s_2)$ holds $s_1 = s_2$,
- (iii) for every elements s_1, s_2 of $\text{Sts } Q$ there exists an element s of $\text{Sts } Q$ such that for every element A of $\text{Obs } Q$ and for every E there exists a real number t such that $0 \leq t$ and $t \leq 1$ and $\text{Meas}(A, s)(E) = t \cdot \text{Meas}(A, s_1)(E) + (1 - t) \cdot \text{Meas}(A, s_2)(E)$.

Next we state a proposition

- (5) Q is a quantum mechanics if and only if the following conditions are satisfied:
 - (i) for all A_1, A_2 such that for every s holds $\text{Meas}(A_1, s) = \text{Meas}(A_2, s)$ holds $A_1 = A_2$,
 - (ii) for all s_1, s_2 such that for every A holds $\text{Meas}(A, s_1) = \text{Meas}(A, s_2)$ holds $s_1 = s_2$,
 - (iii) for every s_1, s_2 there exists s such that for every A, E there exists a real number t such that $0 \leq t$ and $t \leq 1$ and $\text{Meas}(A, s)(E) = t \cdot \text{Meas}(A, s_1)(E) + (1 - t) \cdot \text{Meas}(A, s_2)(E)$.

We follow the rules: Q denotes a quantum mechanics, A, A_1, A_2 denote elements of $\text{Obs } Q$, and s, s_1, s_2 denote elements of $\text{Sts } Q$. We now state three propositions:

- (6) If for every s holds $\text{Meas}(A_1, s) = \text{Meas}(A_2, s)$, then $A_1 = A_2$.
- (7) If for every A holds $\text{Meas}(A, s_1) = \text{Meas}(A, s_2)$, then $s_1 = s_2$.
- (8) For every s_1, s_2 there exists s such that for every A, E there exists a real number t such that $0 \leq t$ and $t \leq 1$ and $\text{Meas}(A, s)(E) = t \cdot \text{Meas}(A, s_1)(E) + (1 - t) \cdot \text{Meas}(A, s_2)(E)$.

We consider POI structures which are systems

\langle a carrier, an ordering, an involution \rangle

where the carrier is a set, the ordering is a relation on the carrier, and the involution is a function from the carrier into the carrier. In the sequel x_1 will denote an element of X_1 , Ord will denote a relation on X_1 , and Inv will denote a function from X_1 into X_1 . Let us consider X_1 . A POI structure is said to be a poset with involution over X_1 if:

the carrier of it = X_1 .

One can prove the following proposition

(9) For every poset W with involution over X_1 holds the carrier of $W = X_1$.

Let us consider X_1, Ord, Inv . The functor $LOG(Ord, Inv)$ yielding a poset with involution over X_1 , is defined by:

$$LOG(Ord, Inv) = \langle X_1, Ord, Inv \rangle.$$

Next we state a proposition

(10) $LOG(Ord, Inv) = \langle X_1, Ord, Inv \rangle$.

Let us consider X_1, Inv . We say that Inv is an involution in X_1 if and only if:

$$Inv(Inv(x_1)) = x_1.$$

We now state a proposition

(11) Inv is an involution in X_1 if and only if for every x_1 holds

$$Inv(Inv(x_1)) = x_1 .$$

Let us consider X_1 , and let W be a poset with involution over X_1 . We say that W is a quantum logic on X_1 if and only if:

there exists a relation Ord on X_1 and there exists a function Inv from X_1 into X_1 such that $W = LOG(Ord, Inv)$ and Ord partially orders X_1 and Inv is an involution in X_1 and for all elements x, y of X_1 such that $\langle x, y \rangle \in Ord$ holds $\langle Inv(y), Inv(x) \rangle \in Ord$.

Next we state a proposition

(12) Let W be a poset with involution over X_1 . Then W is a quantum logic on X_1 if and only if there exists a relation Ord on X_1 and there exists a function Inv from X_1 into X_1 such that $W = LOG(Ord, Inv)$ and Ord partially orders X_1 and Inv is an involution in X_1 and for all elements x, y of X_1 such that $\langle x, y \rangle \in Ord$ holds $\langle Inv(y), Inv(x) \rangle \in Ord$.

Let us consider Q . The functor $Prop Q$ yielding a non-empty set, is defined by:

$$Prop Q = \{ \text{Obs } Q, \text{ the Borel sets} \}.$$

The following proposition is true

(13) $Prop Q = \{ \text{Obs } Q, \text{ the Borel sets} \}$.

In the sequel p, q, r, p_1, q_1 are elements of $Prop Q$. Let us consider Q, p . Then p_1 is an element of $Obs Q$. Then p_2 is an event of the Borel sets.

The following propositions are true:

(14) $p = \langle p_1, p_2 \rangle$.

(15) $(E^c)^c = E$.

(16) For every E such that $E = p_2^c$ holds

$$\text{Meas}(p_1, s)(p_2) = 1 - \text{Meas}(p_1, s)(E) .$$

Let us consider Q, p . The functor $\neg p$ yields an element of $Prop Q$ and is defined as follows:

$$\neg p = \langle p_1, p_2^c \rangle.$$

The following proposition is true

$$(17) \quad \neg p = \langle p_1, p_2^c \rangle.$$

Let us consider Q, p, q . The predicate $p \vdash q$ is defined by:
for every s holds $\text{Meas}(p_1, s)(p_2) \leq \text{Meas}(q_1, s)(q_2)$.

We now state a proposition

$$(18) \quad p \vdash q \text{ if and only if for every } s \text{ holds } \text{Meas}(p_1, s)(p_2) \leq \text{Meas}(q_1, s)(q_2).$$

Let us consider Q, p, q . The predicate $p \equiv q$ is defined as follows:
 $p \vdash q$ and $q \vdash p$.

One can prove the following propositions:

$$(19) \quad p \equiv q \text{ if and only if } p \vdash q \text{ and } q \vdash p.$$

$$(20) \quad p \equiv q \text{ if and only if for every } s \text{ holds } \text{Meas}(p_1, s)(p_2) = \text{Meas}(q_1, s)(q_2).$$

$$(21) \quad p \vdash p.$$

$$(22) \quad \text{If } p \vdash q \text{ and } q \vdash r, \text{ then } p \vdash r.$$

$$(23) \quad p \equiv p.$$

$$(24) \quad \text{If } p \equiv q, \text{ then } q \equiv p.$$

$$(25) \quad \text{If } p \equiv q \text{ and } q \equiv r, \text{ then } p \equiv r.$$

$$(26) \quad (\neg p)_1 = p_1 \text{ and } (\neg p)_2 = p_2^c.$$

$$(27) \quad \neg(\neg p) = p.$$

$$(28) \quad \text{If } p \vdash q, \text{ then } \neg q \vdash \neg p.$$

Let us consider Q . The functor $\text{PropRel } Q$ yields an equivalence relation of $\text{Prop } Q$ and is defined as follows:

$$\langle p, q \rangle \in \text{PropRel } Q \text{ if and only if } p \equiv q.$$

We now state a proposition

$$(29) \quad \langle p, q \rangle \in \text{PropRel } Q \text{ if and only if } p \equiv q.$$

In the sequel B, C will denote subsets of $\text{Prop } Q$. Next we state a proposition

$$(30) \quad \text{For all } B, C \text{ such that } B \in \text{Classes}(\text{PropRel } Q) \text{ and } C \in \text{Classes}(\text{PropRel } Q) \\ \text{for all elements } a, b, c, d \text{ of } \text{Prop } Q \text{ such that } a \in B \text{ and } b \in B \text{ and } c \in C \\ \text{and } d \in C \text{ and } a \vdash c \text{ holds } b \vdash d.$$

Let us consider Q . The functor $\text{OrdRel } Q$ yielding a relation on $\text{Classes}(\text{PropRel } Q)$,

is defined as follows:

$$\langle B, C \rangle \in \text{OrdRel } Q \text{ if and only if } B \in \text{Classes}(\text{PropRel } Q) \text{ and } C \in \text{Classes}(\text{PropRel } Q)$$

and for all p, q such that $p \in B$ and $q \in C$ holds $p \vdash q$.

Next we state four propositions:

$$(31) \quad \langle B, C \rangle \in \text{OrdRel } Q \text{ if and only if } B \in \text{Classes}(\text{PropRel } Q) \text{ and } C \in \text{Classes}(\text{PropRel } Q) \\ \text{and for all } p, q \text{ such that } p \in B \text{ and } q \in C \text{ holds } p \vdash q.$$

$$(32) \quad p \vdash q \text{ if and only if } \langle [p]_{\text{PropRel } Q}, [q]_{\text{PropRel } Q} \rangle \in \text{OrdRel } Q.$$

- (33) For all B, C such that $B \in \text{Classes}(\text{PropRel } Q)$ and $C \in \text{Classes}(\text{PropRel } Q)$
for all p_1, q_1 such that $p_1 \in B$ and $q_1 \in B$ and $\neg p_1 \in C$ holds $\neg q_1 \in C$.
- (34) For all B, C such that $B \in \text{Classes}(\text{PropRel } Q)$ and $C \in \text{Classes}(\text{PropRel } Q)$
for all p, q such that $\neg p \in C$ and $\neg q \in C$ and $p \in B$ holds $q \in B$.

Let us consider Q . The functor $\text{InvRel } Q$ yielding a function from $\text{Classes}(\text{PropRel } Q)$

into $\text{Classes}(\text{PropRel } Q)$, is defined by:

$$(\text{InvRel } Q)([p]_{\text{PropRel } Q}) = [\neg p]_{\text{PropRel } Q}.$$

One can prove the following two propositions:

- (35) $(\text{InvRel } Q)([p]_{\text{PropRel } Q}) = [\neg p]_{\text{PropRel } Q}$.
- (36) For every Q holds $\text{LOG}(\text{OrdRel } Q, \text{InvRel } Q)$ is a quantum logic on $\text{Classes}(\text{PropRel } Q)$.

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