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Interpretation and Satisfiability in the First Order Logic

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Summary. The main notion discussed is satisfiability. Interpretation and some auxiliary concepts are also introduced.

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The articles [6], [3], [1], [5], [4], [2], and [7] provide the notation and terminology for this paper. In the sequel i, k are natural numbers and A, D are non-empty sets. Let us consider A. The functor $\boldsymbol{V}(A)$ yields a non-empty set of functions and is defined by:

$\boldsymbol{V}(A) = A^{\mathrm{BoundVar}}$.

The following propositions are true:

(1) $\boldsymbol{V}(A) = A^{\mathrm{BoundVar}}$.

(2) For an arbitrary x such that x is an element of $\boldsymbol{V}(A)$ holds x is a function from BoundVar into A.

Let us consider A. Then $\boldsymbol{V}(A)$ is a non-empty set of functions from BoundVar to A.

In the sequel x, y will be bound variables and v, v_1 will be elements of $\boldsymbol{V}(A)$. Let us consider A, v, x. Then $v(x)$ is an element of A.

We now define two new functors. Let us consider A, and let p be an element of $Boolean^A$. The functor $\neg p$ yields an element of $Boolean^A$ and is defined by:

for every element x of A holds $(\neg p)(x) = \neg(p(x))$.

Let q be an element of $Boolean^A$. The functor $p \wedge q$ yielding an element of $Boolean^A$ is defined as follows:

for every element x of A holds $(p \wedge q)(x) = (p(x)) \wedge (q(x))$.

We now state two propositions:

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(4)[2] For every element p of $Boolean^A$ and for every element x of A holds $\neg p(x) = \neg(p(x))$.

(5) For all elements p, q of $Boolean^A$ and for every element x of A holds $p \wedge q(x) = (p(x)) \wedge (q(x))$.

Let us consider A, and let f be an element of $Boolean^{\boldsymbol{V}(A)}$, and let us consider v. Then $f(v)$ is an element of $Boolean$.

Let us consider A, x, and let p be an element of $Boolean^{\boldsymbol{V}(A)}$. The functor $\bigwedge_x p$ yields an element of $Boolean^{\boldsymbol{V}(A)}$ and is defined as follows:

for every v holds $(\bigwedge_x p)(v) = Boolean(false \notin \{p(v') : \bigwedge_y [x \neq y \Rightarrow v'(y) = v(y)]\})$.

Next we state three propositions:

(6) For all x, v and for every element p of $Boolean^{\boldsymbol{V}(A)}$ holds $(\bigwedge_x p)(v) = Boolean(false \notin \{p(v') : \bigwedge[x \neq y \Rightarrow v'(y) = v(y)]\})$.

(7) For every element p of $Boolean^{\boldsymbol{V}(A)}$ holds $(\bigwedge_x p)(v) = false$ if and only if there exists v_1 such that $p(v_1) = false$ and for every y such that $x \neq y$ holds $v_1(y) = v(y)$.

(8) For every element p of $Boolean^{\boldsymbol{V}(A)}$ holds $(\bigwedge_x p)(v) = true$ if and only if for every v_1 such that for every y such that $x \neq y$ holds $v_1(y) = v(y)$ holds $p(v_1) = true$.

In the sequel ll is a variables list of k. Let us consider A, v, k, ll. The functor $ll[v]$ yielding a finite sequence of elements of A is defined as follows:

$\mathrm{len}(ll[v]) = k$ and for every i such that $1 \leq i$ and $i \leq k$ holds $(ll[v])(i) = v(ll(i))$.

We now state the proposition

(9) For all v, k, ll holds $\mathrm{len}(ll[v]) = k$ and for every natural number i such that $1 \leq i$ and $i \leq k$ holds $ll[v](i) = v(ll(i))$.

Let us consider A, k, ll, and let r be an element of $\mathrm{Rel}(A)$. The functor $ll\epsilon r$ yields an element of $Boolean^{\boldsymbol{V}(A)}$ and is defined by:

for every element v of $\boldsymbol{V}(A)$ holds if $ll[v] \in r$, then $(ll\epsilon r)(v) = true$ but if $ll[v] \notin r$, then $(ll\epsilon r)(v) = false$.

Next we state the proposition

(10) For all k, ll, v and for every element r of $\mathrm{Rel}(A)$ holds if $ll[v] \in r$, then $ll\epsilon r(v) = true$ but if $ll[v] \notin r$, then $ll\epsilon r(v) = false$.

Let us consider A, and let F be a function from $\mathrm{WFF}_{\mathrm{CQC}}$ into $Boolean^{\boldsymbol{V}(A)}$, and let p be an element of $\mathrm{WFF}_{\mathrm{CQC}}$. Then $F(p)$ is an element of $Boolean^{\boldsymbol{V}(A)}$.

Let us consider D. A function from PredSym into $\mathrm{Rel}(D)$ is called an interpretation of D if:

for every element P of PredSym and for every element r of $\mathrm{Rel}(D)$ such that $\mathrm{it}(P) = r$ holds $r = \varnothing_D$ or $\mathrm{Arity}(P) = \mathrm{Arity}(r)$.

[2]The proposition (3) became obvious.

Next we state two propositions:

(11) For every non-empty set D and for every function F from PredSym into Rel(D) such that for every element P of PredSym and for every element r of Rel(D) such that $F(P) = r$ holds $r = \varnothing_D$ or Arity(P) = Arity(r) holds F is an interpretation of D.

(12) For every D and for every interpretation J of D and for every element P of PredSym and for every element r of Rel(D) such that $J(P) = r$ holds $r = \varnothing_D$ or Arity(P) = Arity(r).

Let us consider A, and let J be an interpretation of A, and let p be an element of PredSym. Then $J(p)$ is a set.

For simplicity we adopt the following rules: p, q, t will be elements of $\mathrm{WFF}_{\mathrm{CQC}}$, J will be an interpretation of A, P will be a k-ary predicate symbol, and r will be an element of Rel(A). Let us consider A, k, J, P. Then $J(P)$ is an element of Rel(A).

Let us consider A, J, p. The functor Valid(p, J) yielding an element of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ is defined by:

there exists a function F from $\mathrm{WFF}_{\mathrm{CQC}}$ into $\mathit{Boolean}^{\boldsymbol{V}(A)}$ such that Valid(p, J) = $F(p)$ and for all elements p, q of $\mathrm{WFF}_{\mathrm{CQC}}$ and for every bound variable x and for every natural number k and for every variables list ll of k and for every k-ary predicate symbol P and for all elements p', q' of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ such that $p' = F(p)$ and $q' = F(q)$ holds

$F(\mathrm{VERUM}) = \boldsymbol{V}(A) \longmapsto \mathit{true}$

and $F(P[ll]) = ll\epsilon(J(P))$ and $F(\neg p) = \neg p'$ and $F(p \wedge q) = p' \wedge q'$ and $F(\forall_x p) = \bigwedge_x p'$.

We now state a number of propositions:

(13) Valid(VERUM, J) = $\boldsymbol{V}(A) \longmapsto \mathit{true}$.

(14) Valid(VERUM, J)(v) = *true*.

(15) Valid($P[ll], J$) = $ll\epsilon(J(P))$.

(16) If $p = P[ll]$ and $r = J(P)$, then $ll[v] \in r$ if and only if Valid(p, J)(v) = *true*.

(17) If $p = P[ll]$ and $r = J(P)$, then $ll[v] \notin r$ if and only if Valid(p, J)(v) = *false*.

(18) If $p = P[ll]$ and $r = J(P)$, then $ll[v] \notin r$ if and only if Valid(p, J)(v) = *false*.

(19) Valid($\neg p, J$) = $\neg$ Valid(p, J).

(20) Valid($\neg p, J$)(v) = $\neg$(Valid(p, J)(v)).

(21) Valid($p \wedge q, J$) = Valid(p, J) $\wedge$ Valid(q, J).

(22) Valid($p \wedge q, J$)(v) = (Valid(p, J)(v)) $\wedge$ (Valid(q, J)(v)).

(23) Valid($\forall_x p, J$) = $\bigwedge_x$ Valid(p, J).

(24) Valid($p \wedge \neg p, J$)(v) = *false*.

(25) Valid($\neg(p \wedge \neg p), J$)(v) = *true*.

Let us consider A, p, J, v. The predicate $J, v \models p$ is defined by:
$\mathrm{Valid}(p, J)(v) = \mathit{true}$.

The following propositions are true:

(26) $J, v \models p$ if and only if $\mathrm{Valid}(p, J)(v) = \mathit{true}$.

(27) $J, v \models P[ll]$ if and only if $ll\epsilon(J(P))(v) = \mathit{true}$.

(28) $J, v \models \neg p$ if and only if $J, v \not\models p$.

(29) $J, v \models p \wedge q$ if and only if $J, v \models p$ and $J, v \models q$.

(30) $J, v \models \forall_x p$ if and only if $(\bigwedge_x \mathrm{Valid}(p, J))(v) = \mathit{true}$.

(31) $J, v \models \forall_x p$ if and only if for every v_1 such that for every y such that $x \neq y$ holds $v_1(y) = v(y)$ holds $\mathrm{Valid}(p, J)(v_1) = \mathit{true}$.

(32) $\mathrm{Valid}(\neg(\neg p), J) = \mathrm{Valid}(p, J)$.

(33) $\mathrm{Valid}(p \wedge p, J) = \mathrm{Valid}(p, J)$.

(34) $\mathrm{Valid}(p \wedge p, J)(v) = \mathrm{Valid}(p, J)(v)$.

(35) $J, v \models p \Rightarrow q$ if and only if $\mathrm{Valid}(p, J)(v) = \mathit{false}$ or $\mathrm{Valid}(q, J)(v) = \mathit{true}$.

(36) $J, v \models p \Rightarrow q$ if and only if if $J, v \models p$, then $J, v \models q$.

(37) For every element p of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ such that $(\bigwedge_x p)(v) = \mathit{true}$ holds $p(v) = \mathit{true}$.

Let us consider A, J, p. The predicate $J \models p$ is defined by:
for every v holds $J, v \models p$.

One can prove the following proposition

(38) $J \models p$ if and only if for every v holds $J, v \models p$.

In the sequel w denotes an element of $\boldsymbol{V}(A)$. The scheme *Lambda_Val* deals with a non-empty set $\mathcal{A}$, a bound variable $\mathcal{B}$, a bound variable $\mathcal{C}$, an element $\mathcal{D}$ of $\boldsymbol{V}(\mathcal{A})$, and an element $\mathcal{E}$ of $\boldsymbol{V}(\mathcal{A})$ and states that:

there exists an element v of $\boldsymbol{V}(\mathcal{A})$ such that for every bound variable x such that $x \neq \mathcal{B}$ holds $v(x) = \mathcal{D}(x)$ and $v(\mathcal{B}) = \mathcal{E}(\mathcal{C})$
for all values of the parameters.

One can prove the following three propositions:

(39) If $x \notin \mathrm{snb}(p)$, then for all v, w such that for every y such that $x \neq y$ holds $w(y) = v(y)$ holds $\mathrm{Valid}(p, J)(v) = \mathrm{Valid}(p, J)(w)$.

(40) If $J, v \models p$ and $x \notin \mathrm{snb}(p)$, then for every w such that for every y such that $x \neq y$ holds $w(y) = v(y)$ holds $J, w \models p$.

(41) $J, v \models \forall_x p$ if and only if for every w such that for every y such that $x \neq y$ holds $w(y) = v(y)$ holds $J, w \models p$.

In the sequel s' will be a formula. We now state a number of propositions:

(42) If $x \neq y$ and $p = s'(x)$ and $q = s'(y)$, then for every v such that $v(x) = v(y)$ holds $\mathrm{Valid}(p, J)(v) = \mathrm{Valid}(q, J)(v)$.

(43) If $x \neq y$ and $x \notin \mathrm{snb}(s')$, then $x \notin \mathrm{snb}(s'(y))$.

(44) $J, v \models \mathrm{VERUM}$.

(45) $J, v \models p \wedge q \Rightarrow q \wedge p$.

(46) $J, v \models (\neg p \Rightarrow p) \Rightarrow p$.
(47) $J, v \models p \Rightarrow (\neg p \Rightarrow q)$.
(48) $J, v \models (p \Rightarrow q) \Rightarrow (\neg(q \wedge t) \Rightarrow \neg(p \wedge t))$.
(49) If $J, v \models p$ and $J, v \models p \Rightarrow q$, then $J, v \models q$.
(50) $J, v \models (\forall_x p) \Rightarrow p$.
(51) $J \models$ VERUM.
(52) $J \models p \wedge q \Rightarrow q \wedge p$.
(53) $J \models (\neg p \Rightarrow p) \Rightarrow p$.
(54) $J \models p \Rightarrow (\neg p \Rightarrow q)$.
(55) $J \models (p \Rightarrow q) \Rightarrow (\neg(q \wedge t) \Rightarrow \neg(p \wedge t))$.
(56) If $J \models p$ and $J \models p \Rightarrow q$, then $J \models q$.
(57) $J \models (\forall_x p) \Rightarrow p$.
(58) If $J \models p \Rightarrow q$ and $x \notin \text{snb}(p)$, then $J \models p \Rightarrow (\forall_x q)$.
(59) For every formula s such that $p = s(x)$ and $q = s(y)$ and $x \notin \text{snb}(s)$ and $J \models p$ holds $J \models q$.

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