Classes of Conjugation. Normal Subgroups

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Summary. Theorems that were not proved in [8] and in [9] are discussed. In the article we define notion of conjugation for elements, subsets and subgroups of a group. We define the classes of conjugation. Normal subgroups of a group and normalizer of a subset of a group or of a subgroup are introduced. We also define the set of all subgroups of a group. An auxiliary theorem that belongs rather to [1] is proved.

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The papers [3], [10], [5], [2], [8], [9], [6], [4], and [7] provide the notation and terminology for this paper. For simplicity we follow a convention: \( x, y \) are arbitrary, \( X \) denotes a set, \( G \) denotes a group, \( a, b, c, d, g, h \) denote elements of \( G \), \( A, B, C, D \) denote subsets of \( G \), \( H, H_1, H_2, H_3 \) denote subgroups of \( G \), \( n \) denotes a natural number, and \( i \) denotes an integer. Next we state a number of propositions:

1. \((a \cdot b) \cdot b^{-1} = a \) and \((a \cdot b^{-1}) \cdot b = a \) and \((b^{-1} \cdot b) \cdot a = a \) and \((b \cdot b^{-1}) \cdot a = a \) and \(a \cdot (b \cdot b^{-1}) = a \) and \(a \cdot (b^{-1} \cdot b) = a \) and \(b^{-1} \cdot (b \cdot a) = a \) and \(b \cdot (b^{-1} \cdot a) = a \).
2. \( G \) is an Abelian group if and only if the operation of \( G \) is commutative.
3. \( \{1\}_G \) is an Abelian group.
4. If \( A \subseteq B \) and \( C \subseteq D \), then \( A \cdot C \subseteq B \cdot D \).
5. If \( A \subseteq B \), then \( a \cdot A \subseteq a \cdot B \) and \( A \cdot a \subseteq B \cdot a \).
6. If \( H_1 \) is a subgroup of \( H_2 \), then \( a \cdot H_1 \subseteq a \cdot H_2 \) and \( H_1 \cdot a \subseteq H_2 \cdot a \).
7. \( a \cdot H = \{a\} \cdot H \).
8. \( H \cdot a = H \cdot \{a\} \).
9. \((a \cdot A) \cdot H = a \cdot (A \cdot H) \).
10. \((A \cdot a) \cdot H = A \cdot (a \cdot H) \).
11. \((a \cdot H) \cdot A = a \cdot (H \cdot A) \).
12. \((A \cdot H) \cdot a = A \cdot (H \cdot a) \).

\(^1\)Supported by RPBP.III-24.C1
(13) \((H \cdot a) \cdot A = H \cdot (a \cdot A)\).
(14) \((H \cdot A) \cdot a = H \cdot (A \cdot a)\).
(15) \((H_1 \cdot a) \cdot H_2 = H_1 \cdot (a \cdot H_2)\).

Let us consider \(G\). The functor \(\text{SubGr} \, G\) yielding a non-empty set is defined by:

(Def.1) \(x \in \text{SubGr} \, G\) if and only if \(x\) is a subgroup of \(G\).

In the sequel \(D\) denotes a non-empty set. Next we state four propositions:

(16) If for every \(x\) holds \(x \in D\) if and only if \(x\) is a subgroup of \(G\), then \(D = \text{SubGr} \, G\).
(17) \(x \in \text{SubGr} \, G\) if and only if \(x\) is a subgroup of \(G\).
(18) \(G \in \text{SubGr} \, G\).
(19) If \(G\) is finite, then \(\text{SubGr} \, G\) is finite.

Let us consider \(G, a, b\). The functor \(a \cdot b\) yielding an element of \(G\) is defined as follows:

(Def.2) \(a \cdot b = (b^{-1} \cdot a) \cdot b\).

One can prove the following propositions:

(20) \(a \cdot b = (b^{-1} \cdot a) \cdot b\) and \(a \cdot b = b^{-1} \cdot (a \cdot b)\).
(21) If \(a^g = b^g\), then \(a = b\).
(22) \((1_G)^a = 1_G\).
(23) If \(a \cdot b = 1_G\), then \(a = 1_G\).
(24) \(a^{1_G} = a\).
(25) \(a^a = a\).
(26) \((a^a)^{-1} = a\) and \((a^{-1})^a = a^{-1}\).
(27) \(a \cdot b = a\) if and only if \(a \cdot b = b \cdot a\).
(28) \((a \cdot b)^g = a^g \cdot b^g\).
(29) \((a^g)^h = a^{gh}\).
(30) \(((a^b)^{-1})^a = a\) and \(((a^b)^{-1})^b = a\).
(31) \(a^b = c\) if and only if \(a \cdot b = b \cdot a\).
(32) \((a^{-1})^b = (a^b)^{-1}\).
(33) \((a^n)^b = (a^b)^n\).
(34) \((a^i)^b = (a^b)^i\).
(35) If \(G\) is an Abelian group, then \(a \cdot b = a\).
(36) If for all \(a, b\) holds \(a \cdot b = a\), then \(G\) is an Abelian group.

Let us consider \(G, A, B\). The functor \(A^B\) yielding a subset of \(G\) is defined as follows:

(Def.3) \(A^B = \{g^h : g \in A \land h \in B\}\).

We now state a number of propositions:

(37) \(A^B = \{g^h : g \in A \land h \in B\}\).
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(38) $x \in A^B$ if and only if there exist $g, h$ such that $x = g^h$ and $g \in A$ and $h \in B$.

(39) $A^B \neq \emptyset$ if and only if $A \neq \emptyset$ and $B \neq \emptyset$.

(40) $A^B \subseteq (B^{-1} \cdot A) \cdot B$.

(41) $(A \cdot B)^C \subseteq A^C \cdot B^C$.

(42) $(A^B)^C = A^{B \cdot C}$.

(43) $(A^{-1})^B = (A^B)^{-1}$.

(44) $\{a\}^B = \{a^b\}$.

(45) $\{a\}^{B,c} = \{a^b, a^c\}$.

(46) $\{a, b\}^c = \{a^c, b^c\}$.

(47) $\{a, b\}^{c,d} = \{a^c, a^d, b^c, b^d\}$.

We now define two new functors. Let us consider $G, A, g$. The functor $A^g$ yields a subset of $G$ and is defined as follows:

(Def.4) $A^g = A^{\{g\}}$.

The functor $g^A$ yields a subset of $G$ and is defined by:

(Def.5) $g^A = \{g\}^A$.

Next we state a number of propositions:

(48) $A^g = A^{\{g\}}$.

(49) $g^A = \{g\}^A$.

(50) $x \in A^g$ if and only if there exists $h$ such that $x = h^g$ and $h \in A$.

(51) $x \in g^A$ if and only if there exists $h$ such that $x = g^h$ and $h \in A$.

(52) $g^A \subseteq (A^{-1} \cdot g) \cdot A$.

(53) $(A^B)^g = A^{B \cdot g}$.

(54) $(A^g)^B = A^g \cdot B$.

(55) $(g^A)^B = g^{A \cdot B}$.

(56) $(A^a)^b = A^{a \cdot b}$.

(57) $(a^A)^b = a^{A \cdot b}$.

(58) $(a^b)^A = a^{b \cdot A}$.

(59) $A^g = (g^{-1} \cdot A) \cdot g$.

(60) $(A \cdot B)^a \subseteq A^a \cdot B^a$.

(61) $A^{1_G} = A$.

(62) If $A \neq \emptyset$, then $(1_G)^A = \{1_G\}$.

(63) $((A^a)^a)^{-1} = A$ and $((A^a)^{-1})^a = A$.

(64) $A = B^g$ if and only if $B = (A^g)^{-1}$.

(65) $G$ is an Abelian group if and only if for all $A, B$ such that $B \neq \emptyset$ holds $A^B = A$.

(66) $G$ is an Abelian group if and only if for all $A, g$ holds $A^g = A$.

(67) $G$ is an Abelian group if and only if for all $A, g$ such that $A \neq \emptyset$ holds $g^A = \{g\}$. 
Let us consider $G$, $H$, $a$. The functor $H^a$ yielding a subgroup of $G$ is defined by:

(Def.6) the carrier of $H^a = \overline{a}^i$.

The following propositions are true:

(68) If the carrier of $H_1 = \overline{a}^i$, then $H_1 = H^a$.
(69) The carrier of $H^a = \overline{H}^a$.
(70) $x \in H^a$ if and only if there exists $g$ such that $x = g^a$ and $g \in H$.
(71) The carrier of $H^a = (a^{-1} \cdot H) \cdot a$.
(72) $(H^a)^b = H^{a \cdot b}$.
(73) $H_1^G = H$.
(74) $((H^a)^a)^{-1} = H$ and $((H^a)^{-1})^a = H$.
(75) $H_1 = H_2^a$ if and only if $H_2 = (H_1^a)^{-1}$.
(76) $(H_1 \cap H_2)^a = H_1^a \cap H_2^a$.
(77) $\text{Ord}(H) = \text{Ord}(H^a)$.
(78) $H$ is finite if and only if $H^a$ is finite.
(79) If $H$ is finite, then $\text{ord}(H) = \text{ord}(H^a)$.
(80) $\{1\}_G^a = \{1\}_G$.
(81) If $H^a = \{1\}_G$, then $H = \{1\}_G$.
(82) $\Omega_{G^a} = G$.
(83) If $H^a = G$, then $H = G$.
(84) $|\ast : H| = |\ast : H^a|$.
(85) If the left cosets of $H$ is finite, then $|\ast : H|_N = |\ast : H^a|_N$.
(86) If $G$ is an Abelian group, then for all $H$, $a$ holds $H^a = H$.

Let us consider $G$, $a$, $b$. We say that $a$ and $b$ are conjugated if and only if:

(Def.7) there exists $g$ such that $a = b^g$.

We now state several propositions:

(87) $a$ and $b$ are conjugated if and only if there exists $g$ such that $a = b^g$.
(88) $a$ and $b$ are conjugated if and only if there exists $g$ such that $b = a^g$.
(89) $a$ and $a$ are conjugated.
(90) If $a$ and $b$ are conjugated, then $b$ and $a$ are conjugated.
(91) If $a$ and $b$ are conjugated and $b$ and $c$ are conjugated, then $a$ and $c$ are conjugated.
(92) If $a$ and $1_G$ are conjugated or $1_G$ and $a$ are conjugated, then $a = 1_G$.
(93) $a^{1_G} = \{b : a \text{ and } b \text{ are conjugated}\}$.

Let us consider $G$, $a$. The functor $a^\ast$ yielding a subset of $G$ is defined by:

(Def.8) $a^\ast = a^{1_G}$.

We now state several propositions:

(94) $a^\ast = a^{1_G}$. 
(95) \( x \in a^* \) if and only if there exists \( b \) such that \( b = x \) and \( a \) and \( b \) are conjugated.

(96) \( a \in b^* \) if and only if \( a \) and \( b \) are conjugated.

(97) \( a^g \in a^* \).

(98) \( a \in a^* \).

(99) If \( a \in b^* \), then \( b \in a^* \).

(100) \( a^* = b^* \) if and only if \( a^* \) meets \( b^* \).

(101) \( a^* = \{1_G\} \) if and only if \( a = 1_G \).

(102) \( a^*: A = A \cdot a^* \).

Let us consider \( G, A, B \). We say that \( A \) and \( B \) are conjugated if and only if:

(Def.9) there exists \( g \) such that \( A = B^g \).

We now state several propositions:

(103) \( A \) and \( B \) are conjugated if and only if there exists \( g \) such that \( A = B^g \).

(104) \( A \) and \( B \) are conjugated if and only if there exists \( g \) such that \( B = A^g \).

(105) \( A \) and \( A \) are conjugated.

(106) If \( A \) and \( B \) are conjugated, then \( B \) and \( A \) are conjugated.

(107) If \( A \) and \( B \) are conjugated and \( B \) and \( C \) are conjugated, then \( A \) and \( C \) are conjugated.

(108) \( \{a\} \) and \( \{b\} \) are conjugated if and only if \( a \) and \( b \) are conjugated.

(109) If \( A \) and \( H_1 \) are conjugated, then there exists \( H_2 \) such that the carrier of \( H_2 \) is \( A \).

Let us consider \( G, A \). The functor \( A^* \) yielding a family of subsets of the carrier of \( G \) is defined as follows:

(Def.10) \( A^* = \{B : A \) and \( B \) are conjugated \} \).

The following propositions are true:

(110) \( A^* = \{B : A \) and \( B \) are conjugated \} \).

(111) \( x \in A^* \) if and only if there exists \( B \) such that \( x = B \) and \( A \) and \( B \) are conjugated.

(112) If \( x \in A^* \), then \( x \) is a subset of \( G \).

(113) \( A \in B^* \) if and only if \( A \) and \( B \) are conjugated.

(114) \( A^g \in A^* \).

(115) \( A \in A^* \).

(116) If \( A \in B^* \), then \( B \in A^* \).

(117) \( A^* = B^* \) if and only if \( A^* \) meets \( B^* \).

(118) \( \{a\}^* = \{\{b\} : b \in a^* \} \).

(119) If \( G \) is finite, then \( A^* \) is finite.

Let us consider \( G, H_1, H_2 \). We say that \( H_1 \) and \( H_2 \) are conjugated if and only if:

(Def.11) there exists \( g \) such that \( H_1 = H_2^g \).
The following propositions are true:

(120) \( H_1 \) and \( H_2 \) are conjugated if and only if there exists \( g \) such that \( H_1 = H_2^g \).

(121) \( H_1 \) and \( H_2 \) are conjugated if and only if there exists \( g \) such that \( H_2 = H_1^g \).

(122) \( H_1 \) and \( H_1 \) are conjugated.

(123) If \( H_1 \) and \( H_2 \) are conjugated, then \( H_2 \) and \( H_1 \) are conjugated.

(124) If \( H_1 \) and \( H_2 \) are conjugated and \( H_2 \) and \( H_3 \) are conjugated, then \( H_1 \) and \( H_3 \) are conjugated.

In the sequel \( L \) denotes a subset of \( \text{SubGr} G \). Let us consider \( G, H \). The functor \( H^* \) yielding a subset of \( \text{SubGr} G \) is defined as follows:

(Def.12) \( x \in H^* \) if and only if there exists \( H_1 \) such that \( x = H_1 \) and \( H_1 \) and \( H \) are conjugated.

One can prove the following propositions:

(125) If for every \( x \) holds \( x \in L \) if and only if there exists \( H \) such that \( x = H \) and \( H_1 \) and \( H \) are conjugated, then \( L = H_1^* \).

(126) \( x \in H_1^* \) if and only if there exists \( H_2 \) such that \( x = H_2 \) and \( H_1 \) and \( H_2 \) are conjugated.

(127) If \( x \in H^* \), then \( x \) is a subgroup of \( G \).

(128) \( H_1 \in H_2^* \) if and only if \( H_1 \) and \( H_2 \) are conjugated.

(129) \( H^0 \in H^* \).

(130) \( H \in H^* \).

(131) If \( H_1 \in H_2^* \), then \( H_2 \in H_1^* \).

(132) \( H_1^* = H_2^* \) if and only if \( H_1^* \) meets \( H_2^* \).

(133) If \( G \) is finite, then \( H^* \) is finite.

(134) \( H_1 \) and \( H_2 \) are conjugated if and only if \( \overline{H_1} \) and \( \overline{H_2} \) are conjugated.

Let us consider \( G \). A subgroup of \( G \) is called a normal subgroup of \( G \) if:

(Def.13) for every \( a \) holds \( a^a = a \).

One can prove the following proposition

(135) If for every \( a \) holds \( H = H^a \), then \( H \) is a normal subgroup of \( G \).

In the sequel \( N, N_1, N_2 \) will denote ha normal subgroups of \( G \). We now state a number of propositions:

(136) \( N^a = N \).

(137) \( \{1\}_G \) is a normal subgroup of \( G \) and \( \Omega_G \) is a normal subgroup of \( G \).

(138) \( N_1 \cap N_2 \) is a normal subgroup of \( G \).

(139) If \( G \) is an Abelian group, then \( H \) is a normal subgroup of \( G \).

(140) \( H \) is a normal subgroup of \( G \) if and only if for every \( a \) holds \( a \cdot H = H \cdot a \).

(141) \( H \) is a normal subgroup of \( G \) if and only if for every \( a \) holds \( a \cdot H \subseteq H \cdot a \).

(142) \( H \) is a normal subgroup of \( G \) if and only if for every \( a \) holds \( H \cdot a \subseteq a \cdot H \).

(143) \( H \) is a normal subgroup of \( G \) if and only if for every \( A \) holds \( A \cdot H = H \cdot A \).
(144) $H$ is a normal subgroup of $G$ if and only if for every $a$ holds $H$ is a subgroup of $H^a$.

(145) $H$ is a normal subgroup of $G$ if and only if for every $a$ holds $H^a$ is a subgroup of $H$.

(146) $H$ is a normal subgroup of $G$ if and only if $H^* = \{H\}$.

(147) $H$ is a normal subgroup of $G$ if and only if for every $a$ such that $a \in H$ holds $a^* \subseteq H$.

(148) $N_1 \cdot N_2 = N_2 \cdot N_1$.

(149) There exists $N$ such that the carrier of $N = N_1 \cdot N_2$.

(150) The left cosets of $N$ = the right cosets of $N$.

(151) If the left cosets of $H$ is finite and $|\cdot : H|_N = 2$, then $H$ is a normal subgroup of $G$.

Let us consider $G$, $A$. The functor $N(A)$ yielding a subgroup of $G$ is defined by:

(Def.14) the carrier of $N(A) = \{h : A^h = A\}$.

We now state several propositions:

(152) If the carrier of $H = \{h : A^h = A\}$, then $H = N(A)$.

(153) The carrier of $N(A) = \{h : A^h = A\}$.

(154) $x \in N(A)$ if and only if there exists $h$ such that $x = h$ and $A^h = A$.

(155) $\overline{A^*} = |\cdot : N(A)|$.

(156) If $A^*$ is finite or the left cosets of $N(A)$ is finite, then $\text{card } A^* = |\cdot : N(A)|_N$.

(157) $\overline{a^*} = |\cdot : N(\{a\})|_N$.

(158) If $a^*$ is finite or the left cosets of $N(\{a\})$ is finite, then $\text{card } a^* = |\cdot : N(\{a\})|_N$.

Let us consider $G$, $H$. The functor $N(H)$ yields a subgroup of $G$ and is defined as follows:

(Def.15) $N(H) = N(\overline{H})$.

We now state several propositions:

(159) $N(H) = N(\overline{H})$.

(160) $x \in N(H)$ if and only if there exists $h$ such that $x = h$ and $H^h = H$.

(161) $\overline{N(H)} = |\cdot : N(H)|_N$.

(162) If $H^*$ is finite or the left cosets of $N(H)$ is finite, then $\text{card } H^* = |\cdot : N(H)|_N$.

(163) $H$ is a normal subgroup of $G$ if and only if $N(H) = G$.

(164) $N(\{1\}_G) = G$.

(165) $N(\Omega_G) = G$.

(166) If $X$ is finite and $\text{card } X = 2$, then there exist $x$, $y$ such that $x \neq y$ and $X = \{x, y\}$.
References


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