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Hessenberg Theorem [1]

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Summary. We prove the Hessenberg theorem which states that every Pappian projective space is Desarguesian.

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The terminology and notation used in this paper are introduced in the following articles: [7], [1], [2], [3], [4], [5], and [6]. We follow a convention: P_1 denotes a projective space defined in terms of collinearity and a, a', a_1, a_2, a_3, b, b', b_1, b_2, c, c', c_1, c_3, d, d', e, o, p, p_1, p_2, p_3, q, q_1, q_2, q_3, r, s, x, y, z denote elements of the points of P_1. One can prove the following propositions:

(1) If a, b and c are collinear, then b, a and c are collinear.

(2) If a, b and c are collinear, then a, c and b are collinear.

(3) If a, b and c are collinear, then b, c and a are collinear and c, a and b are collinear and b, a and c are collinear and a, c and b are collinear and c, b and a are collinear.

(4) If $a \neq b$ and a, b and c are collinear and a, b and d are collinear, then a, c and d are collinear.

(5) If $p \neq q$ and a, b and p are collinear and a, b and q are collinear and p, q and r are collinear, then a, b and r are collinear.

(6) If $p \neq q$, then there exists r such that p, q and r are not collinear.

(7) There exist q, r such that p, q and r are not collinear.

(8) If a, b and c are not collinear and a, b and b' are collinear and $a \neq b'$, then a, b' and c are not collinear.

(9) If a, b and c are not collinear and a, b and d are collinear and a, c and d are collinear, then $a = d$.

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(10) If o, a and d are not collinear and o, d and d' are collinear and a, d and s are collinear and $d \neq d'$ and a', d' and s are collinear and o, a and a' are collinear and $o \neq a'$, then $s \neq d$.

(11) If a, b and c are not collinear and a, b and b' are collinear and a, c and c' are collinear and $a \neq b'$, then $b' \neq c'$.

(12) If a_1, a_2 and a_3 are not collinear and a_1, a_2 and c_3 are collinear and a_2, a_3 and c_1 are collinear and a_1, a_3 and z are collinear and c_1, c_3 and z are collinear and $c_3 \neq a_1$ and $c_3 \neq a_2$ and $c_1 \neq a_2$ and $c_1 \neq a_3$, then $a_1 \neq z$ and $a_3 \neq z$.

(13) If a, b and c are not collinear and a, b and d are collinear and c, e and d are collinear and $e \neq c$ and $d \neq a$, then e, a and c are not collinear.

(14) If p_1, p_2 and q_1 are not collinear and p_1, p_2 and q_2 are collinear and q_1, q_2 and q_3 are collinear and $p_1 \neq q_2$ and $q_2 \neq q_3$, then p_2, p_1 and q_3 are not collinear.

(15) If p_1, p_2 and q_1 are not collinear and p_1, p_2 and p_3 are collinear and q_1, q_2 and p_3 are collinear and $p_3 \neq q_2$ and $p_2 \neq p_3$, then p_3, p_2 and q_2 are not collinear.

(16) If p_1, p_2 and q_1 are not collinear and p_1, p_2 and p_3 are collinear and q_1, q_2 and p_1 are collinear and $p_1 \neq p_3$ and $p_1 \neq q_2$, then p_3, p_1 and q_2 are not collinear.

(17) If $a_1 \neq a_2$ and $b_1 \neq b_2$ and b_1, b_2 and x are collinear and b_1, b_2 and y are collinear and a_1, a_2 and x are collinear and a_1, a_2 and y are collinear and a_1, a_2 and b_1 are not collinear, then $x = y$.

(19)[2] If o, a_1 and a_2 are not collinear and o, a_1 and b_1 are collinear and o, a_2 and b_2 are collinear and $o \neq b_1$ and $o \neq b_2$, then o, b_1 and b_2 are not collinear.

We follow a convention: P_1 denotes a Pappian projective plane defined in terms of collinearity and a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 denote elements of the points of P_1. We now state two propositions:

(20) Suppose that

(i) $p_2 \neq p_3$,
(ii) $p_1 \neq p_3$,
(iii) $q_2 \neq q_3$,
(iv) $q_1 \neq q_2$,
(v) $q_1 \neq q_3$,
(vi) p_1, p_2 and q_1 are not collinear,
(vii) p_1, p_2 and p_3 are collinear,
(viii) q_1, q_2 and q_3 are collinear,
(ix) p_1, q_2 and r_3 are collinear,
(x) q_1, p_2 and r_3 are collinear,
(xi) p_1, q_3 and r_2 are collinear,
(xii) p_3, q_1 and r_2 are collinear,

[2]The proposition (18) was either repeated or obvious.

(xiii) p_2, q_3 and r_1 are collinear,
(xiv) p_3, q_2 and r_1 are collinear.
Then r_1, r_2 and r_3 are collinear.

(21) Suppose that
(i) $o \neq b_1$,
(ii) $a_1 \neq b_1$,
(iii) $o \neq b_2$,
(iv) $a_2 \neq b_2$,
(v) $o \neq b_3$,
(vi) $a_3 \neq b_3$,
(vii) o, a_1 and a_2 are not collinear,
(viii) o, a_1 and a_3 are not collinear,
(ix) o, a_2 and a_3 are not collinear,
(x) a_1, a_2 and c_3 are collinear,
(xi) b_1, b_2 and c_3 are collinear,
(xii) a_2, a_3 and c_1 are collinear,
(xiii) b_2, b_3 and c_1 are collinear,
(xiv) a_1, a_3 and c_2 are collinear,
(xv) b_1, b_3 and c_2 are collinear,
(xvi) o, a_1 and b_1 are collinear,
(xvii) o, a_2 and b_2 are collinear,
(xviii) o, a_3 and b_3 are collinear.
Then c_1, c_2 and c_3 are collinear.

We see that the Pappian projective plane defined in terms of collinearity is a Desarguesian projective plane defined in terms of collinearity.

We see that the Pappian projective space defined in terms of collinearity is a Desarguesian projective space defined in terms of collinearity.

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