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König's Lemma [1]

Grzegorz Bancerek
Warsaw University
Białystok

Summary. A contiuation of [5]. The notions of finite-order trees, succesors of an element of a tree, and chains, levels and branches of a tree are introduced. Those notions are used to formalize König's Lemma which claims that there is a infinite branch of a finite-order tree if the tree has arbitrary long finite chains. Besides, the concept of decorated trees is introduced and some concepts dealing with trees are applied to decorated trees.

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The articles [12], [7], [10], [4], [6], [9], [2], [1], [3], [8], [11], [13], and [5] provide the notation and terminology for this paper. For simplicity we adopt the following rules: x, y are arbitrary, W, W_1, W_2 denote trees, w denotes an element of W, X denotes a set, f, f_1, f_2 denote functions, D, D' denote non-empty sets, k, k_1, k_2, m, n denote natural numbers, v, v_1, v_2 denote finite sequences, and p, q, r denote finite sequences of elements of $\mathbb{N}$. The following propositions are true:

(1) For all v_1, v_2, v such that $v_1 \preceq v$ and $v_2 \preceq v$ holds v_1 and v_2 are comparable.

(2) For all v_1, v_2, v such that $v_1 \prec v$ and $v_2 \preceq v$ holds v_1 and v_2 are comparable and v_2 and v_1 are comparable.

(4)[2] If $\operatorname{len} v_1 = k + 1$, then there exist v_2, x such that $v_1 = v_2 \frown \langle x \rangle$ and $\operatorname{len} v_2 = k$.

(5) $(v_1 \frown v_2) \restriction \operatorname{Seg} \operatorname{len} v_1 = v_1$.

(6) $\operatorname{Seg}_{\preceq}(v \frown \langle x \rangle) = \operatorname{Seg}_{\preceq}(v) \cup \{v\}$.

The scheme *TreeStruct_Ind* concerns a tree $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every element t of $\mathcal{A}$ holds $\mathcal{P}[t]$

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[2]The proposition (3) was either repeated or obvious.

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provided the following requirements are met:

- $\mathcal{P}[\varepsilon]$,
- for every element t of $\mathcal{A}$ and for every n such that $\mathcal{P}[t]$ and $t \frown \langle n \rangle \in \mathcal{A}$ holds $\mathcal{P}[t \frown \langle n \rangle]$.

We now state the proposition

(7) If for every p holds $p \in W_1$ if and only if $p \in W_2$, then $W_1 = W_2$.

Let us consider W_1, W_2. Let us note that one can characterize the predicate $W_1 = W_2$ by the following (equivalent) condition:

(Def.1) for every p holds $p \in W_1$ if and only if $p \in W_2$.

One can prove the following propositions:

(8) If $p \in W$, then $W = W(p/(W \restriction p))$.

(9) If $p \in W$ and $q \in W$ and $p \npreceq q$, then $q \in W(p/W_1)$.

(10) If $p \in W$ and $q \in W$ and p and q are not comparable, then $W(p/W_1)(q/W_2) = W(q/W_2)(p/W_1)$.

A tree is finite-order if:

(Def.2) there exists n such that for every element t of it holds $t \frown \langle n \rangle \notin$ it.

We now define three new constructions. Let us consider W. A subset of W is said to be a chain of W if:

(Def.3) for all p, q such that $p \in$ it and $q \in$ it holds p and q are comparable.

A subset of W is called a level of W if:

(Def.4) there exists n such that it $= \{w : \mathrm{len}\, w = n\}$.

Let us consider w. The functor $\mathrm{succ}\, w$ yielding a subset of W is defined by:

(Def.5) $\mathrm{succ}\, w = \{w \frown \langle n \rangle : w \frown \langle n \rangle \in W\}$.

One can prove the following propositions:

(11) For every level L of W holds L is an antichain of prefixes of W.

(12) $\mathrm{succ}\, w$ is an antichain of prefixes of W.

(13) For every antichain A of prefixes of W and for every chain C of W there exists w such that $A \cap C \subseteq \{w\}$.

Let us consider W, n. The functor n_W yielding a level of W is defined by:

(Def.6) $n_W = \{w : \mathrm{len}\, w = n\}$.

We now state several propositions:

(14) $w \frown \langle n \rangle \in \mathrm{succ}\, w$ if and only if $w \frown \langle n \rangle \in W$.

(15) If $w = \varepsilon$, then $1_W = \mathrm{succ}\, w$.

(16) $W = \bigcup\{n_W\}$.

(17) For every finite tree W holds $W = \bigcup\{n_W : n \leq \mathrm{height}\, W\}$.

(18) For every level L of W there exists n such that $L = n_W$.

Now we present three schemes. The scheme *AuxSch* concerns a tree $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\{w : \mathcal{P}[w]\}$, where w ranges over elements of $\mathcal{A}$, is a subset of $\mathcal{A}$

for all values of the parameters.

The scheme *FraenkelCard* concerns a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

$\overline{\overline{\{\mathcal{F}(w) : w \in \mathcal{B}\}}} \leq \overline{\overline{\mathcal{B}}}$, where w ranges over elements of $\mathcal{A}$

for all values of the parameters.

The scheme *FraenkelFinCard* concerns a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

$\mathrm{card}\{\mathcal{F}(w) : w \in \mathcal{B}\} \leq \mathrm{card}\,\mathcal{B}$, where w ranges over elements of $\mathcal{A}$

provided the parameters meet the following requirement:

- $\mathcal{B}$ is finite.

The following four propositions are true:

(19) If W is finite-order, then there exists n such that for every w holds $\mathrm{succ}\,w$ is finite and $\mathrm{card}\,\mathrm{succ}\,w \leq n$.

(20) If W is finite-order, then $\mathrm{succ}\,w$ is finite.

(21) $\emptyset$ is a chain of W.

(22) $\{\varepsilon\}$ is a chain of W.

Let us consider W. A chain of W is said to be a branch of W if:

(Def.7) for every p such that $p \in$ it holds $\mathrm{Seg}_{\preceq}(p) \subseteq$ it and for no p holds $p \in W$ and for every q such that $q \in$ it holds $q \prec p$.

Let us consider W. We see that the branch of W is an non-empty chain of W.

In the sequel C will be a chain of W and B will be a branch of W. The following propositions are true:

(23) If $v_1 \in C$ and $v_2 \in C$, then $v_1 \in \mathrm{Seg}_{\preceq}(v_2)$ or $v_2 \preceq v_1$.

(24) If $v_1 \in C$ and $v_2 \in C$ and $v \preceq v_2$, then $v_1 \in \mathrm{Seg}_{\preceq}(v)$ or $v \preceq v_1$.

(25) If C is finite and $\mathrm{card}\,C > n$, then there exists p such that $p \in C$ and $\mathrm{len}\,p \geq n$.

(26) For every C holds $\{w : \bigvee_p [p \in C \wedge w \preceq p]\}$ is a chain of W.

(27) If $p \preceq q$ and $q \in B$, then $p \in B$.

(28) $\varepsilon \in B$.

(29) If $p \in C$ and $q \in C$ and $\mathrm{len}\,p \leq \mathrm{len}\,q$, then $p \preceq q$.

(30) There exists B such that $C \subseteq B$.

Now we present two schemes. The scheme *FuncExOfMinNat* concerns a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists f such that $\mathrm{dom}\,f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ there exists n such that $f(x) = n$ and $\mathcal{P}[x, n]$ and for every m such that $\mathcal{P}[x, m]$ holds $n \leq m$

provided the following condition is met:

- for every x such that $x \in \mathcal{A}$ there exists n such that $\mathcal{P}[x, n]$.

The scheme *InfiniteChain* concerns a set $\mathcal{A}$, a constant $\mathcal{B}$, a unary predicate $\mathcal{P}$, and a binary predicate $\mathcal{Q}$, and states that:

there exists f such that $\mathrm{dom}\,f = \mathbb{N}$ and $\mathrm{rng}\,f \subseteq \mathcal{A}$ and $f(0) = \mathcal{B}$ and for every k holds $\mathcal{Q}[f(k), f(k+1)]$ and $\mathcal{P}[f(k)]$

provided the parameters meet the following conditions:

- $\mathcal{B} \in \mathcal{A}$ and $\mathcal{P}[\mathcal{B}]$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{P}[x]$ there exists y such that $y \in \mathcal{A}$ and $\mathcal{Q}[x, y]$ and $\mathcal{P}[y]$.

The following two propositions are true:

(31) For every tree T such that for every n there exists a chain C of T such that C is finite and $\operatorname{card} C = n$ and for every element t of T holds $\operatorname{succ} t$ is finite there exists a chain B of T such that B is not finite.

(32) For every finite-order tree T such that for every n there exists a chain C of T such that C is finite and $\operatorname{card} C = n$ there exists a chain B of T such that B is not finite.

A function is said to be a decorated tree if:

(Def.8) dom it is a tree.

In the sequel T, T_1, T_2 are decorated trees. Let us consider T. Then $\operatorname{dom} T$ is a tree.

Let us consider D. A decorated tree is said to be a tree decorated by D if:

(Def.9) $\operatorname{rng} \mathrm{it} \subseteq D$.

Let D be a non-empty set, and let T be a tree decorated by D, and let t be an element of $\operatorname{dom} T$. Then $T(t)$ is an element of D.

One can prove the following proposition

(33) If $\operatorname{dom} T_1 = \operatorname{dom} T_2$ and for every p such that $p \in \operatorname{dom} T_1$ holds $T_1(p) = T_2(p)$, then $T_1 = T_2$.

Now we present two schemes. The scheme *DTreeEx* concerns a tree $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists T such that $\operatorname{dom} T = \mathcal{A}$ and for every p such that $p \in \mathcal{A}$ holds $\mathcal{P}[p, T(p)]$

provided the following condition is satisfied:

- for every p such that $p \in \mathcal{A}$ there exists x such that $\mathcal{P}[p, x]$.

The scheme *DTreeLambda* deals with a tree $\mathcal{A}$ and a unary functor $\mathcal{F}$ and states that:

there exists T such that $\operatorname{dom} T = \mathcal{A}$ and for every p such that $p \in \mathcal{A}$ holds $T(p) = \mathcal{F}(p)$

for all values of the parameters.

We now define two new functors. Let us consider T. The functor $\operatorname{Leaves} T$ yielding a set is defined by:

(Def.10) $\operatorname{Leaves} T = T^{\circ} \operatorname{Leaves} \operatorname{dom} T$.

Let us consider p. The functor $T \restriction p$ yielding a decorated tree is defined by:

(Def.11) $\operatorname{dom}(T \restriction p) = \operatorname{dom} T \restriction p$ and for every q such that $q \in \operatorname{dom} T \restriction p$ holds $(T \restriction p)(q) = T(p \frown q)$.

The following proposition is true

(34) If $p \in \operatorname{dom} T$, then $\operatorname{rng}(T \restriction p) \subseteq \operatorname{rng} T$.

Let us consider D, and let T be a tree decorated by D. Then $\mathrm{Leaves}\,T$ is a subset of D. Let p be an element of $\mathrm{dom}\,T$. Then $T \upharpoonright p$ is a tree decorated by D.

Let us consider T, p, T_1. Let us assume that $p \in \mathrm{dom}\,T$. The functor $T(p/T_1)$ yielding a decorated tree is defined by the conditions (Def.12).

(Def.12) (i) $\mathrm{dom}(T(p/T_1)) = (\mathrm{dom}\,T)(p/\mathrm{dom}\,T_1)$,

(ii) for every q such that $q \in (\mathrm{dom}\,T)(p/\mathrm{dom}\,T_1)$ holds $p \not\preceq q$ and $T(p/T_1)(q) = T(q)$ or there exists r such that $r \in \mathrm{dom}\,T_1$ and $q = p \frown r$ and $T(p/T_1)(q) = T_1(r)$.

Let us consider W, x. Then $W \longmapsto x$ is a decorated tree.

Let D be a non-empty set, and let us consider W, and let d be an element of D. Then $W \longmapsto d$ is a tree decorated by D.

Next we state four propositions:

(35) If for every x such that $x \in D$ holds x is a tree, then $\bigcup D$ is a tree.

(36) If for every x such that $x \in X$ holds x is a function and for all f_1, f_2 such that $f_1 \in X$ and $f_2 \in X$ holds $\mathrm{graph}\,f_1 \subseteq \mathrm{graph}\,f_2$ or $\mathrm{graph}\,f_2 \subseteq \mathrm{graph}\,f_1$, then $\bigcup X$ is a function.

(37) If for every x such that $x \in D$ holds x is a decorated tree and for all T_1, T_2 such that $T_1 \in D$ and $T_2 \in D$ holds $\mathrm{graph}\,T_1 \subseteq \mathrm{graph}\,T_2$ or $\mathrm{graph}\,T_2 \subseteq \mathrm{graph}\,T_1$, then $\bigcup D$ is a decorated tree.

(38) If for every x such that $x \in D'$ holds x is a tree decorated by D and for all T_1, T_2 such that $T_1 \in D'$ and $T_2 \in D'$ holds $\mathrm{graph}\,T_1 \subseteq \mathrm{graph}\,T_2$ or $\mathrm{graph}\,T_2 \subseteq \mathrm{graph}\,T_1$, then $\bigcup D'$ is a tree decorated by D.

Now we present two schemes. The scheme *DTreeStructEx* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a function $\mathcal{C}$ from $[: \mathcal{A}, \mathbb{N} :]$ into $\mathcal{A}$ and states that:

there exists a tree T decorated by $\mathcal{A}$ such that $T(\varepsilon) = \mathcal{B}$ and for every element t of $\mathrm{dom}\,T$ holds $\mathrm{succ}\,t = \{t \frown \langle k \rangle : k \in \mathcal{F}(T(t))\}$ and for all n, x such that $x = T(t)$ and $n \in \mathcal{F}(x)$ holds $T(t \frown \langle n \rangle) = \mathcal{C}(\langle x, n \rangle)$

provided the following condition is satisfied:

- for every element d of $\mathcal{A}$ and for all k_1, k_2 such that $k_1 \leq k_2$ and $k_2 \in \mathcal{F}(d)$ holds $k_1 \in \mathcal{F}(d)$.

The scheme *DTreeStructFinEx* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a natural number, and a function $\mathcal{C}$ from $[: \mathcal{A}, \mathbb{N} :]$ into $\mathcal{A}$ and states that:

there exists a tree T decorated by $\mathcal{A}$ such that $T(\varepsilon) = \mathcal{B}$ and for every element t of $\mathrm{dom}\,T$ holds $\mathrm{succ}\,t = \{t \frown \langle k \rangle : k < \mathcal{F}(T(t))\}$ and for all n, x such that $x = T(t)$ and $n < \mathcal{F}(x)$ holds $T(t \frown \langle n \rangle) = \mathcal{C}(\langle x, n \rangle)$

for all values of the parameters.

References

[1] Grzegorz Bancerek. Cardinal arithmetics. *Formalized Mathematics*, 1(**3**):543–547, 1990.
[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[3] Grzegorz Bancerek. Countable sets and Hessenberg's theorem. *Formalized Mathematics*, 2(**1**):65–69, 1991.
[4] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[5] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.
[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[11] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[13] Wojciech A. Trybulec and Grzegorz Bancerek. Kuratowski - Zorn lemma. *Formalized Mathematics*, 1(**2**):387–393, 1990.

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