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The Topological Space $\mathcal{E}^2_{\rm T}$. Arcs, Line Segments and Special Polygonal Arcs

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Summary. The notions of arc and line segment are introduced in two-dimensional topological real space $\mathcal{E}^2_{\rm T}$. Some basic theorems for these notions are proved. Using line segments, the notion of special polygonal arc is defined. It has been shown that any special polygonal arc is homeomorphic to unit interval $\mathbb{I}$. The notion of unit square $\square_{\mathcal{E}^2_{\rm T}}$ has been also introduced and some facts about it have been proved.

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The articles [22], [21], [13], [1], [24], [20], [6], [7], [18], [4], [8], [15], [23], [17], [25], [11], [16], [9], [19], [2], [5], [14], [3], [10], and [12] provide the notation and terminology for this paper. In the sequel l_1 will denote a real number and i, j, n will denote natural numbers. The scheme *Fraenkel_Alt* concerns a non-empty set $\mathcal{A}$, and two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

$\{v : \mathcal{P}[v] \vee \mathcal{Q}[v]\} = \{v_1 : \mathcal{P}[v_1]\} \cup \{v_2 : \mathcal{Q}[v_2]\}$, where v_2 ranges over elements of $\mathcal{A}$, and v_1 ranges over elements of $\mathcal{A}$, and v ranges over elements of $\mathcal{A}$

for all values of the parameters.

In the sequel d_1, d_2, d_3 will be arbitrary. We now state the proposition

(2)[2] $\langle d_1, d_2, d_3 \rangle$ is one-to-one if and only if $d_1 \neq d_2$ and $d_2 \neq d_3$ and $d_1 \neq d_3$.

In the sequel D denotes a non-empty set and p denotes a finite sequence of elements of D. Let us consider D, p, n. The functor $p \restriction n$ yielding a finite sequence of elements of D is defined by:

(Def.1) $p \restriction n = p \restriction \operatorname{Seg} n$.

One can prove the following proposition

[1]The article was written during my work at Shinshu University, 1991.
[2]The proposition (1) has been removed.

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(3) If $n \leq \operatorname{len} p$, then $\operatorname{len}(p \restriction n) = n$.

Let us consider T. A finite sequence of elements of T is a finite sequence of elements of the carrier of T.

We adopt the following convention: p, p_1, p_2, q, q_1, q_2 will be points of $\mathcal{E}^2_{\mathrm{T}}$ and P, Q, P_1, P_2 will be subsets of $\mathcal{E}^2_{\mathrm{T}}$. Let us consider p_1, p_2, P. We say that P is an arc from p_1 to p_2 if and only if:

(Def.2) $P \neq \emptyset$ and there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ such that f is a homeomorphism and $f(0) = p_1$ and $f(1) = p_2$.

One can prove the following two propositions:

(4) If P is an arc from p_1 to p_2, then $p_1 \in P$ and $p_2 \in P$.

(5) If P is an arc from p_1 to p_2 and Q is an arc from p_2 to q_1 and $P \cap Q = \{p_2\}$, then $P \cup Q$ is an arc from p_1 to q_1.

The subset $\square_{\mathcal{E}^2}$ of $\mathcal{E}^2_{\mathrm{T}}$ is defined by the condition (Def.3).

(Def.3) $\square_{\mathcal{E}^2} = \{p : p_{\mathbf{1}} = 0 \wedge p_{\mathbf{2}} \leq 1 \wedge p_{\mathbf{2}} \geq 0 \ \vee \ p_{\mathbf{1}} \leq 1 \wedge p_{\mathbf{1}} \geq 0 \wedge p_{\mathbf{2}} = 1 \ \vee \ p_{\mathbf{1}} \leq 1 \wedge p_{\mathbf{1}} \geq 0 \wedge p_{\mathbf{2}} = 0 \ \vee \ p_{\mathbf{1}} = 1 \wedge p_{\mathbf{2}} \leq 1 \wedge p_{\mathbf{2}} \geq 0\}$.

Let us consider p_1, p_2. The functor $\mathcal{L}(p_1, p_2)$ yielding a non-empty subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def.4) $\mathcal{L}(p_1, p_2) = \{p : \bigvee_{l_1} [0 \leq l_1 \wedge l_1 \leq 1 \wedge p = (1 - l_1) \cdot p_1 + l_1 \cdot p_2]\}$.

Next we state a number of propositions:

(6) $p_1 \in \mathcal{L}(p_1, p_2)$ and $p_2 \in \mathcal{L}(p_1, p_2)$.

(7) $\mathcal{L}(p, p) = \{p\}$.

(8) $\mathcal{L}(p_1, p_2) = \mathcal{L}(p_2, p_1)$.

(9) If $p_{1\mathbf{1}} \leq p_{2\mathbf{1}}$ and $p \in \mathcal{L}(p_1, p_2)$, then $p_{1\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq p_{2\mathbf{1}}$.

(10) If $p_{1\mathbf{2}} \leq p_{2\mathbf{2}}$ and $p \in \mathcal{L}(p_1, p_2)$, then $p_{1\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq p_{2\mathbf{2}}$.

(11) If $p \in \mathcal{L}(p_1, p_2)$, then $\mathcal{L}(p_1, p_2) = \mathcal{L}(p_1, p) \cup \mathcal{L}(p, p_2)$.

(12) If $q_1 \in \mathcal{L}(p_1, p_2)$ and $q_2 \in \mathcal{L}(p_1, p_2)$, then $\mathcal{L}(q_1, q_2) \subseteq \mathcal{L}(p_1, p_2)$.

(13) If $p \in \mathcal{L}(p_1, p_2)$ and $q \in \mathcal{L}(p_1, p_2)$, then $\mathcal{L}(p_1, p_2) = \mathcal{L}(p_1, p) \cup \mathcal{L}(p, q) \cup \mathcal{L}(q, p_2)$.

(14) If $p \in \mathcal{L}(p_1, p_2)$, then $\mathcal{L}(p_1, p) \cap \mathcal{L}(p, p_2) = \{p\}$.

(15) If $p_1 \neq p_2$, then $\mathcal{L}(p_1, p_2)$ is an arc from p_1 to p_2.

(16) If P is an arc from p_1 to p_2 and $P \cap \mathcal{L}(p_2, q_1) = \{p_2\}$, then $P \cup \mathcal{L}(p_2, q_1)$ is an arc from p_1 to q_1.

(17) If P is an arc from p_2 to p_1 and $\mathcal{L}(q_1, p_2) \cap P = \{p_2\}$, then $\mathcal{L}(q_1, p_2) \cup P$ is an arc from q_1 to p_1.

(18) If $p_1 \neq p_2$ or $p_2 \neq q_1$ but $\mathcal{L}(p_1, p_2) \cap \mathcal{L}(p_2, q_1) = \{p_2\}$, then $\mathcal{L}(p_1, p_2) \cup \mathcal{L}(p_2, q_1)$ is an arc from p_1 to q_1.

(19) (i) $\mathcal{L}([0, 0], [0, 1]) = \{p_1 : p_{1\mathbf{1}} = 0 \wedge p_{1\mathbf{2}} \leq 1 \wedge p_{1\mathbf{2}} \geq 0\}$,

(ii) $\mathcal{L}([0, 1], [1, 1]) = \{p_2 : p_{2\mathbf{1}} \leq 1 \wedge p_{2\mathbf{1}} \geq 0 \wedge p_{2\mathbf{2}} = 1\}$,

(iii) $\mathcal{L}([0, 0], [1, 0]) = \{q_1 : q_{1\mathbf{1}} \leq 1 \wedge q_{1\mathbf{1}} \geq 0 \wedge q_{1\mathbf{2}} = 0\}$,

(iv) $\mathcal{L}([1, 0], [1, 1]) = \{q_2 : q_{2\mathbf{1}} = 1 \wedge q_{2\mathbf{2}} \leq 1 \wedge q_{2\mathbf{2}} \geq 0\}$.

(20) $\square_{\mathcal{E}^2} = \mathcal{L}([0,0],[0,1]) \cup \mathcal{L}([0,1],[1,1]) \cup (\mathcal{L}([0,0],[1,0]) \cup \mathcal{L}([1,0],[1,1]))$.

(21) $\mathcal{L}([0,0],[0,1]) \cap \mathcal{L}([0,1],[1,1]) = \{[0,1]\}$.

(22) $\mathcal{L}([0,0],[1,0]) \cap \mathcal{L}([1,0],[1,1]) = \{[1,0]\}$.

(23) $\mathcal{L}([0,0],[0,1]) \cap \mathcal{L}([0,0],[1,0]) = \{[0,0]\}$.

(24) $\mathcal{L}([0,1],[1,1]) \cap \mathcal{L}([1,0],[1,1]) = \{[1,1]\}$.

(25) $\mathcal{L}([0,0],[1,0]) \cap \mathcal{L}([0,1],[1,1]) = \emptyset$.

(26) $\mathcal{L}([0,0],[0,1]) \cap \mathcal{L}([1,0],[1,1]) = \emptyset$.

In the sequel f, f_1, f_2, h will be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Let us consider f, i, j. The functor $\mathcal{L}(f,i,j)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def.5) (i) for all p_1, p_2 such that $p_1 = f(i)$ and $p_2 = f(j)$ holds $\mathcal{L}(f,i,j) = \mathcal{L}(p_1,p_2)$ if $i \in \mathrm{Seg}\,\mathrm{len}\,f$ and $j \in \mathrm{Seg}\,\mathrm{len}\,f$,

(ii) $\mathcal{L}(f,i,j) = \emptyset$, otherwise.

The following proposition is true

(27) If $i \in \mathrm{Seg}\,\mathrm{len}\,f$ and $j \in \mathrm{Seg}\,\mathrm{len}\,f$, then $f(i) \in \mathcal{L}(f,i,j)$ and $f(j) \in \mathcal{L}(f,i,j)$.

Let us consider f. The functor $\widetilde{\mathcal{L}}(f)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def.6) $\widetilde{\mathcal{L}}(f) = \bigcup\{\mathcal{L}(f,i,i+1) : 1 \leq i \wedge i \leq \mathrm{len}\,f - 1\}$.

One can prove the following propositions:

(28) $\mathrm{len}\,f = 0$ or $\mathrm{len}\,f = 1$ if and only if $\widetilde{\mathcal{L}}(f) = \emptyset$.

(29) If $\mathrm{len}\,f \geq 2$, then $\widetilde{\mathcal{L}}(f) \neq \emptyset$.

Let us consider f. We say that f is a special sequence if and only if the conditions (Def.7) is satisfied.

(Def.7) (i) f is one-to-one,

(ii) $\mathrm{len}\,f \geq 3$,

(iii) for every i such that $1 \leq i$ and $i \leq \mathrm{len}\,f - 2$ holds $\mathcal{L}(f,i,i+1) \cap \mathcal{L}(f,i+1,i+2) = \{f(i+1)\}$,

(iv) for all i, j such that $i - j > 1$ or $j - i > 1$ holds $\mathcal{L}(f,i,i+1) \cap \mathcal{L}(f,j,j+1) = \emptyset$,

(v) for all i, p_1, p_2 such that $1 \leq i$ and $i \leq \mathrm{len}\,f - 1$ and $p_1 = f(i)$ and $p_2 = f(i+1)$ holds $p_{1\mathbf{1}} = p_{2\mathbf{1}}$ or $p_{1\mathbf{2}} = p_{2\mathbf{2}}$.

The following propositions are true:

(30) There exist f_1, f_2 such that f_1 is a special sequence and f_2 is a special sequence and $\square_{\mathcal{E}^2} = \widetilde{\mathcal{L}}(f_1) \cup \widetilde{\mathcal{L}}(f_2)$ and $\widetilde{\mathcal{L}}(f_1) \cap \widetilde{\mathcal{L}}(f_2) = \{[0,0],[1,1]\}$ and $f_1(1) = [0,0]$ and $f_1(\mathrm{len}\,f_1) = [1,1]$ and $f_2(1) = [0,0]$ and $f_2(\mathrm{len}\,f_2) = [1,1]$.

(31) If h is a special sequence and $P = \widetilde{\mathcal{L}}(h)$, then for all p_1, p_2 such that $p_1 = h(1)$ and $p_2 = h(\mathrm{len}\,h)$ holds P is an arc from p_1 to p_2.

Let us consider P. We say that P is a special polygonal arc if and only if:

(Def.8) there exists f such that f is a special sequence and $P = \widetilde{\mathcal{L}}(f)$.

The following propositions are true:

(32) If P is a special polygonal arc, then $P \neq \emptyset$.

(33) If f is a special sequence, then $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc.

(34) There exist P_1, P_2 such that P_1 is a special polygonal arc and P_2 is a special polygonal arc and $\square_{\mathcal{E}^2} = P_1 \cup P_2$ and $P_1 \cap P_2 = \{[0,0],[1,1]\}$.

(35) If P is a special polygonal arc, then there exist p_1, p_2 such that P is an arc from p_1 to p_2.

(36) If P is a special polygonal arc, then there exists a map f from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction P$ such that f is a homeomorphism.

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