

Properties of Caratheodor's Measure

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Summary. The paper contains definitions and basic properties of Caratheodor's measure, with values in $\overline{\mathbb{R}}$, the enlarged set of real numbers, where $\overline{\mathbb{R}}$ denotes set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ - by [14]. The article includes the text being a continuation of the paper [3]. Caratheodor's theorem and some theorems concerning basic properties of Caratheodor's measure are proved. The work is the sixth part of the series of articles concerning the Lebesgue measure theory.

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The terminology and notation used in this paper have been introduced in the following papers: [16], [15], [10], [11], [8], [9], [1], [13], [2], [12], [4], [5], [7], [6], [3], and [17]. One can prove the following propositions:

- (1) For all *Real numbers* x, y, z such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$ and $0_{\overline{\mathbb{R}}} \leq z$ holds $(x + y) + z = x + (y + z)$.
- (2) For all *Real numbers* x, y, z such that $x \neq -\infty$ and $x \neq +\infty$ holds $y + x \leq z$ if and only if $y \leq z - x$.
- (3) For all *Real numbers* x, y such that $0_{\overline{\mathbb{R}}} \leq x$ and $0_{\overline{\mathbb{R}}} \leq y$ holds $x + y = y + x$.
- (4) For every set X and for every σ -field S of subsets of X and for every function F from \mathbb{N} into S and for every element A of S and for every function G from \mathbb{N} into S such that for every element n of \mathbb{N} holds $G(n) = A \cap F(n)$ holds $\bigcup \text{rng } G = A \cap \bigcup \text{rng } F$.
- (5) Let X be a set. Let S be a σ -field of subsets of X . Let F be a function from \mathbb{N} into S . Let G be a function from \mathbb{N} into S . Suppose $G(0) = F(0)$ and for every element n of \mathbb{N} holds $G(n + 1) = F(n + 1) \cup G(n)$. Then for every function H from \mathbb{N} into S such that $H(0) = F(0)$ and for every element n of \mathbb{N} holds $H(n + 1) = F(n + 1) \setminus G(n)$ holds $\bigcup \text{rng } F = \bigcup \text{rng } H$.
- (6) For every set X holds 2^X is a σ -field of subsets of X .

Let X be a set, and let F be a function from \mathbb{N} into 2^X . Then $\text{rng } F$ is a non-empty family of subsets of X . Let A be a non-empty family of subsets of X . Then $\bigcup A$ is an element of 2^X . Let F be a function from 2^X into $\overline{\mathbb{R}}$. We say that F is non-negative if and only if:

(Def.1) for every element A of 2^X holds $0_{\overline{\mathbb{R}}} \leq F(A)$.

Let F be a function from \mathbb{N} into 2^X , and let M be a function from 2^X into $\overline{\mathbb{R}}$. Then $M \cdot F$ is a function from \mathbb{N} into $\overline{\mathbb{R}}$.

One can prove the following propositions:

- (7) For every set X and for every *Real numbers* a, b there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds if $A = \emptyset$, then $M(A) = a$ but if $A \neq \emptyset$, then $M(A) = b$.
- (8) For every set X there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that for every element A of 2^X holds $M(A) = 0_{\overline{\mathbb{R}}}$.
- (9) For every set X and for every function F from \mathbb{N} into 2^X and for every function M from 2^X into $\overline{\mathbb{R}}$ such that M is non-negative holds $M \cdot F$ is non-negative.
- (10) For every set X and for every function F from \mathbb{N} into 2^X and for every function M from 2^X into $\overline{\mathbb{R}}$ and for every natural number n holds $(M \cdot F)(n) = M(F(n))$.
- (11) Let X be a set. Then there exists a function M from 2^X into $\overline{\mathbb{R}}$ such that M is non-negative and $M(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of 2^X such that $A \subseteq B$ holds $M(A) \leq M(B)$ and for every function F from \mathbb{N} into 2^X holds $M(\bigcup \text{rng } F) \leq \sum(M \cdot F)$.

We now define two new constructions. Let X be a set. A function from 2^X into $\overline{\mathbb{R}}$ is said to be a Caratheodor's measure on X if:

(Def.2) it is non-negative and $it(\emptyset) = 0_{\overline{\mathbb{R}}}$ and for all elements A, B of 2^X such that $A \subseteq B$ holds $it(A) \leq it(B)$ and for every function F from \mathbb{N} into 2^X holds $it(\bigcup \text{rng } F) \leq \sum(it \cdot F)$.

Let C be a Caratheodor's measure on X . The functor $\sigma\text{-Field}(C)$ yielding a non-empty family of subsets of X is defined by:

(Def.3) for every element A of 2^X holds $A \in \sigma\text{-Field}(C)$ if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) \leq C(W \cup Z)$.

The following propositions are true:

- (12) For every set X and for every Caratheodor's measure C on X and for all elements W, Z of 2^X holds $C(W \cup Z) \leq C(W) + C(Z)$.
- (13) For every set X and for every Caratheodor's measure C on X and for all elements W, Z of 2^X holds $C(Z) + C(W) = C(W) + C(Z)$.
- (14) For every set X and for every Caratheodor's measure C on X and for every element A of 2^X holds $A \in \sigma\text{-Field}(C)$ if and only if for all elements W, Z of 2^X such that $W \subseteq A$ and $Z \subseteq X \setminus A$ holds $C(W) + C(Z) = C(W \cup Z)$.

- (15) For every set X and for every Caratheodor's measure C on X and for all elements W, Z of 2^X such that $W \in \sigma\text{-Field}(C)$ and $Z \in \sigma\text{-Field}(C)$ and $Z \cap W = \emptyset$ holds $C(W \cup Z) = C(W) + C(Z)$.
- (16) For every set X and for every Caratheodor's measure C on X and for every set A such that $A \in \sigma\text{-Field}(C)$ holds $X \setminus A \in \sigma\text{-Field}(C)$.
- (17) For every set X and for every Caratheodor's measure C on X and for all sets A, B such that $A \in \sigma\text{-Field}(C)$ and $B \in \sigma\text{-Field}(C)$ holds $A \cup B \in \sigma\text{-Field}(C)$.
- (18) For every set X and for every Caratheodor's measure C on X and for all sets A, B such that $A \in \sigma\text{-Field}(C)$ and $B \in \sigma\text{-Field}(C)$ holds $A \cap B \in \sigma\text{-Field}(C)$.
- (19) For every set X and for every Caratheodor's measure C on X and for all sets A, B such that $A \in \sigma\text{-Field}(C)$ and $B \in \sigma\text{-Field}(C)$ holds $A \setminus B \in \sigma\text{-Field}(C)$.
- (20) For every set X and for every σ -field S of subsets of X and for every function N from \mathbb{N} into S and for every element A of S there exists a function F from \mathbb{N} into S such that for every element n of \mathbb{N} holds $F(n) = A \cap N(n)$.
- (21) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Field}(C)$ is a σ -field of subsets of X .

Let X be a set, and let C be a Caratheodor's measure on X . Then $\sigma\text{-Field}(C)$ is a σ -field of subsets of X . Let S be a σ -field of subsets of X , and let A be a subfamily of S . Then $\bigcup A$ is an element of S . The functor $\sigma\text{-Meas}(C)$ yields a function from $\sigma\text{-Field}(C)$ into $\overline{\mathbb{R}}$ and is defined by:

- (Def.4) for every element A of 2^X such that $A \in \sigma\text{-Field}(C)$ holds $(\sigma\text{-Meas}(C))(A) = C(A)$.

One can prove the following proposition

- (22) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Meas}(C)$ is a measure on $\sigma\text{-Field}(C)$.

Let X be a set, and let C be a Caratheodor's measure on X , and let A be an element of $\sigma\text{-Field}(C)$. Then $C(A)$ is a *Real number*.

One can prove the following proposition

- (23) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Meas}(C)$ is a σ -measure on $\sigma\text{-Field}(C)$.

Let X be a set, and let C be a Caratheodor's measure on X . Then $\sigma\text{-Meas}(C)$ is a σ -measure on $\sigma\text{-Field}(C)$.

The following propositions are true:

- (24) For every set X and for every Caratheodor's measure C on X and for every element A of 2^X such that $C(A) = 0_{\overline{\mathbb{R}}}$ holds $A \in \sigma\text{-Field}(C)$.
- (25) For every set X and for every Caratheodor's measure C on X holds $\sigma\text{-Meas}(C)$ is complete on $\sigma\text{-Field}(C)$.

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