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Homomorphisms of Algebras. Quotient Universal Algebra

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Summary. The first part introduces homomorphisms of universal algebras and their basic properties. The second is concerned with the construction of a quotient universal algebra. The first isomorphism theorem is proved.

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The articles [9], [10], [11], [4], [5], [1], [8], [3], [6], [7], and [2] provide the terminology and notation for this paper.

1. Homomorphisms of Algebras

For simplicity we adopt the following convention: U_1, U_2, U_3 will denote universal algebras, n will denote a natural number, o_1 will denote a operation of U_1, o_2 will denote a operation of U_2, and x, y will be arbitrary.

Let D_1, D_2 be non empty set, let p be a finite sequence of elements of D_1, and let f be a function from D_1 into D_2. Then $f \cdot p$ is a finite sequence of elements of D_2.

The following propositions are true:

(1) Let D_1, D_2 be non empty set, and let p be a finite sequence of elements of D_1, and let f be a function from D_1 into D_2. Then $\mathrm{dom}(f \cdot p) = \mathrm{dom}\, p$ and $\mathrm{len}(f \cdot p) = \mathrm{len}\, p$ and for every n such that $n \in \mathrm{dom}(f \cdot p)$ holds $(f \cdot p)(n) = f(p(n))$.

(2) For every non empty subset B of U_1 such that $B =$ the carrier of U_1 holds $\mathrm{Opers}(U_1, B) = \mathrm{Opers}\, U_1$.

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Let U_1 be a universal algebra. A finite sequence of elements of U_1 is a finite sequence of elements of the carrier of U_1. Let U_2 be a universal algebra. A function from U_1 into U_2 is a function from the carrier of U_1 into the carrier of U_2.

In the sequel a, a_1, a_2 denote finite sequences of elements of U_1 and f denotes a function from U_1 into U_2.

One can prove the following three propositions:

(3) $f \cdot \varepsilon_{(\text{the carrier of } U_1)} = \varepsilon_{(\text{the carrier of } U_2)}$.

(4) $\mathrm{id}_{(\text{the carrier of } U_1)} \cdot a = a$.

(5) Let h_1 be a function from U_1 into U_2, and let h_2 be a function from U_2 into U_3, and let a be a finite sequence of elements of U_1. Then $h_2 \cdot (h_1 \cdot a) = (h_2 \cdot h_1) \cdot a$.

Let us consider U_1, U_2, f. We say that f is a homomorphism of U_1 into U_2 if and only if the conditions (Def.1) are satisfied.

(Def.1) (i) U_1 and U_2 are similar, and

(ii) for every n such that $n \in \operatorname{dom} \operatorname{Opers} U_1$ and for all o_1, o_2 such that $o_1 = (\operatorname{Opers} U_1)(n)$ and $o_2 = (\operatorname{Opers} U_2)(n)$ and for every finite sequence x of elements of U_1 such that $x \in \operatorname{dom} o_1$ holds $f(o_1(x)) = o_2(f \cdot x)$.

Let us consider U_1, U_2, f. We say that f is a monomorphism of U_1 into U_2 if and only if:

(Def.2) f is a homomorphism of U_1 into U_2 and one-to-one.

We say that f is an epimorphism of U_1 onto U_2 if and only if:

(Def.3) f is a homomorphism of U_1 into U_2 and $\operatorname{rng} f =$ the carrier of U_2.

Let us consider U_1, U_2, f. We say that f is an isomorphism of U_1 and U_2 if and only if:

(Def.4) f is a monomorphism of U_1 into U_2 and an epimorphism of U_1 onto U_2.

Let us consider U_1, U_2. We say that U_1 and U_2 are isomorphic if and only if:

(Def.5) There exists f which is an isomorphism of U_1 and U_2.

One can prove the following propositions:

(6) $\mathrm{id}_{(\text{the carrier of } U_1)}$ is a homomorphism of U_1 into U_1.

(7) Let h_1 be a function from U_1 into U_2 and let h_2 be a function from U_2 into U_3. Suppose h_1 is a homomorphism of U_1 into U_2 and h_2 is a homomorphism of U_2 into U_3. Then $h_2 \cdot h_1$ is a homomorphism of U_1 into U_3.

(8) f is an isomorphism of U_1 and U_2 if and only if f is a homomorphism of U_1 into U_2 and $\operatorname{rng} f =$ the carrier of U_2 and f is one-to-one.

(9) If f is an isomorphism of U_1 and U_2, then $\operatorname{dom} f =$ the carrier of U_1 and $\operatorname{rng} f =$ the carrier of U_2.

(10) Let h be a function from U_1 into U_2 and let h_1 be a function from U_2 into U_1. Suppose h is an isomorphism of U_1 and U_2 and $h_1 = h^{-1}$. Then h_1 is a homomorphism of U_2 into U_1.

(11) Let h be a function from U_1 into U_2 and let h_1 be a function from U_2 into U_1. Suppose h is an isomorphism of U_1 and U_2 and $h_1 = h^{-1}$. Then h_1 is an isomorphism of U_2 and U_1.

(12) Let h be a function from U_1 into U_2 and let h_1 be a function from U_2 into U_3. Suppose h is an isomorphism of U_1 and U_2 and h_1 is an isomorphism of U_2 and U_3. Then $h_1 \cdot h$ is an isomorphism of U_1 and U_3.

(13) U_1 and U_1 are isomorphic.

(14) If U_1 and U_2 are isomorphic, then U_2 and U_1 are isomorphic.

(15) If U_1 and U_2 are isomorphic and U_2 and U_3 are isomorphic, then U_1 and U_3 are isomorphic.

Let us consider U_1, U_2, f. Let us assume that f is a homomorphism of U_1 into U_2. The functor $\mathrm{Im}\, f$ yielding a strict subalgebra of U_2 is defined as follows:

(Def.6) The carrier of $\mathrm{Im}\, f = f\,^\circ$ (the carrier of U_1).

Next we state two propositions:

(16) For every function h from U_1 into U_2 such that h is a homomorphism of U_1 into U_2 holds $\mathrm{rng}\, h =$ the carrier of $\mathrm{Im}\, h$.

(17) Let U_2 be a strict universal algebra and let f be a function from U_1 into U_2. Suppose f is a homomorphism of U_1 into U_2. Then f is an epimorphism of U_1 onto U_2 if and only if $\mathrm{Im}\, f = U_2$.

2. Quotient Universal Algebra

Let us consider U_1. A binary relation on U_1 is a binary relation on the carrier of U_1. An equivalence relation of U_1 is an equivalence relation of the carrier of U_1.

Let D be a non empty set and let R be a binary relation on D. The functor $R^{\#}$ yielding a binary relation on D^* is defined by the condition (Def.7).

(Def.7) Let x, y be finite sequences of elements of D. Then $\langle x, y\rangle \in R^{\#}$ if and only if the following conditions are satisfied:

(i) $\mathrm{len}\, x = \mathrm{len}\, y$, and

(ii) for every n such that $n \in \mathrm{dom}\, x$ holds $\langle x(n), y(n)\rangle \in R$.

The following proposition is true

(18) For every non empty set D holds $(\triangle_D)^{\#} = \triangle_{D^*}$.

Let us consider U_1. An equivalence relation of U_1 is said to be a congruence of U_1 if it satisfies the condition (Def.8).

(Def.8) Given n, o_1. Suppose $n \in \mathrm{dom}\, \mathrm{Opers}\, U_1$ and $o_1 = (\mathrm{Opers}\, U_1)(n)$. Let x, y be finite sequences of elements of U_1. If $x \in \mathrm{dom}\, o_1$ and $y \in \mathrm{dom}\, o_1$ and $\langle x, y\rangle \in \mathrm{it}^{\#}$, then $\langle o_1(x), o_1(y)\rangle \in \mathrm{it}$.

Let D be a non empty set and let R be an equivalence relation of D. Then $\mathrm{Classes}\, R$ is a non empty family of subsets of D.

Let D be a non empty set, let R be an equivalence relation of D, let y be a finite sequence of elements of Classes R, and let x be a finite sequence of elements of D. We say that x is a finite sequence of representatives of y if and only if:

(Def.9) $\operatorname{len} x = \operatorname{len} y$ and for every n such that $n \in \operatorname{dom} x$ holds $[x(n)]_R = y(n)$.

We now state the proposition

(19) Let D be a non empty set, and let R be an equivalence relation of D, and let y be a finite sequence of elements of Classes R. Then there exists finite sequence of elements of D which is a finite sequence of representatives of y.

Let U_1 be a universal algebra, let E be a congruence of U_1, and let o be a operation of U_1. The functor $o_{/E}$ yields a homogeneous quasi total non-empty partial function from (Classes E)* to Classes E and is defined by the conditions (Def.10).

(Def.10) (i) $\operatorname{dom}(o_{/E}) = (\text{Classes } E)^{\text{arity}\, o}$, and

(ii) for every finite sequence y of elements of Classes E such that $y \in \operatorname{dom}(o_{/E})$ and for every finite sequence x of elements of the carrier of U_1 such that x is a finite sequence of representatives of y holds $o_{/E}(y) = [o(x)]_E$.

Let us consider U_1, E. The functor $\operatorname{Opers}(U_1)_{/E}$ yields a finite sequence of elements of (Classes E)$^* \dot{\rightarrow}$ Classes E and is defined as follows:

(Def.11) $\operatorname{len}(\operatorname{Opers}((U_1))_{/E}) = \operatorname{len} \operatorname{Opers} U_1$ and for every n such that $n \in \operatorname{dom}(\operatorname{Opers}((U_1))_{/E})$ and for every o_1 such that $(\operatorname{Opers} U_1)(n) = o_1$ holds $\operatorname{Opers}((U_1))_{/E}(n) = (o_1)_{/E}$.

Next we state the proposition

(20) For all U_1, E holds $\langle \text{Classes } E, \operatorname{Opers}((U_1))_{/E} \rangle$ is a strict universal algebra.

Let us consider U_1, E. The functor $U_{1/E}$ yielding a strict universal algebra is defined by:

(Def.12) $(U_1)_{/E} = \langle \text{Classes } E, \operatorname{Opers}((U_1))_{/E} \rangle$.

Let us consider U_1, E. The natural homomorphism of U_1 w.r.t. E yielding a function from U_1 into $(U_1)_{/E}$ is defined as follows:

(Def.13) For every element u of the carrier of U_1 holds (the natural homomorphism of U_1 w.r.t. E)$(u) = [u]_E$.

One can prove the following two propositions:

(21) For all U_1, E holds the natural homomorphism of U_1 w.r.t. E is a homomorphism of U_1 into $(U_1)_{/E}$.

(22) For all U_1, E holds the natural homomorphism of U_1 w.r.t. E is an epimorphism of U_1 onto $(U_1)_{/E}$.

Let us consider U_1, U_2 and let f be a function from U_1 into U_2. Let us assume that f is a homomorphism of U_1 into U_2. The functor $\operatorname{Cng}(f)$ yielding a congruence of U_1 is defined by:

(Def.14) For all elements a, b of the carrier of U_1 holds $\langle a, b\rangle \in \mathrm{Cng}(f)$ iff $f(a) = f(b)$.

Let U_1, U_2 be universal algebras and let f be a function from U_1 into U_2. Let us assume that f is a homomorphism of U_1 into U_2. The functor $\overline{f}$ yielding a function from $(U_1)_{/\,\mathrm{Cng}(f)}$ into U_2 is defined by:

(Def.15) For every element a of the carrier of U_1 holds $(\overline{f})([a]_{\mathrm{Cng}(f)}) = f(a)$.

We now state three propositions:

(23) Suppose f is a homomorphism of U_1 into U_2. Then $\overline{f}$ is a homomorphism of $(U_1)_{/\,\mathrm{Cng}(f)}$ into U_2 and $\overline{f}$ is a monomorphism of $(U_1)_{/\,\mathrm{Cng}(f)}$ into U_2.

(24) If f is an epimorphism of U_1 onto U_2, then $\overline{f}$ is an isomorphism of $(U_1)_{/\,\mathrm{Cng}(f)}$ and U_2.

(25) If f is an epimorphism of U_1 onto U_2, then $(U_1)_{/\,\mathrm{Cng}(f)}$ and U_2 are isomorphic.

References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[2] Ewa Burakowska. Subalgebras of the universal algebra. Lattices of subalgebras. *Formalized Mathematics*, 4(**1**):23–27, 1993.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[7] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Formalized Mathematics*, 3(**2**):251–253, 1992.

[8] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[10] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[11] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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