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Homomorphisms of Lattices, Finite Join and Finite Meet

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The articles [9], [4], [2], [3], [8], [10], [6], [1], [5], and [7] provide the terminology and notation for this paper.

1. Preliminaries

We adopt the following convention: X, X_1, X_2, Y, Z will denote sets and x will be arbitrary.

Next we state three propositions:

(1) If $\bigcup Y \subseteq Z$ and $X \in Y$, then $X \subseteq Z$.

(2) $\bigcup(X \Cap Y) = \bigcup X \cap \bigcup Y$.

(3) Given X. Suppose that

(i) $X \neq \emptyset$, and

(ii) for every Z such that $Z \neq \emptyset$ and $Z \subseteq X$ and for all X_1, X_2 such that $X_1 \in Z$ and $X_2 \in Z$ holds $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$ there exists Y such that $Y \in X$ and for every X_1 such that $X_1 \in Z$ holds $X_1 \subseteq Y$.

Then there exists Y such that $Y \in X$ and for every Z such that $Z \in X$ and $Z \neq Y$ holds $Y \not\subseteq Z$.

2. Lattice Theory

We adopt the following convention: L denotes a lattice, F, H denote filters of L, and p, q, r denote elements of the carrier of L.

One can prove the following propositions:

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(4) $[L)$ is prime.

(5) $F \subseteq [F \cup H)$ and $H \subseteq [F \cup H)$.

(6) If $p \in [[q) \cup F)$, then there exists r such that $r \in F$ and $q \sqcap r \sqsubseteq p$.

We adopt the following rules: L_1, L_2 will be lattices, a_1, b_1 will be elements of the carrier of L_1, and a_2 will be an element of the carrier of L_2.

Let us consider L_1, L_2. A function from the carrier of L_1 into the carrier of L_2 is called a homomorphism from L_1 to L_2 if:

(Def.1) $\mathrm{It}(a_1 \sqcup b_1) = \mathrm{it}(a_1) \sqcup \mathrm{it}(b_1)$ and $\mathrm{it}(a_1 \sqcap b_1) = \mathrm{it}(a_1) \sqcap \mathrm{it}(b_1)$.

In the sequel f is a homomorphism from L_1 to L_2.

We now state the proposition

(7) If $a_1 \sqsubseteq b_1$, then $f(a_1) \sqsubseteq f(b_1)$.

Let us consider L_1, L_2, f. We say that f is monomorphism if and only if:

(Def.2) f is one-to-one.

We say that f is epimorphism if and only if:

(Def.3) $\mathrm{rng}\, f =$ the carrier of L_2.

Next we state two propositions:

(8) If f is monomorphism, then $a_1 \sqsubseteq b_1$ iff $f(a_1) \sqsubseteq f(b_1)$.

(9) If f is epimorphism, then for every a_2 there exists a_1 such that $a_2 = f(a_1)$.

Let us consider L_1, L_2, f. We say that f is isomorphism if and only if:

(Def.4) f is monomorphism and epimorphism.

Let us consider L_1, L_2. We say that L_1 and L_2 are isomorphic if and only if:

(Def.5) There exists f which is isomorphism.

Let us consider L_1, L_2, f. We say that f preserves implication if and only if:

(Def.6) $f(a_1 \Rightarrow b_1) = f(a_1) \Rightarrow f(b_1)$.

We say that f preserves top if and only if:

(Def.7) $f(\top_{(L_1)}) = \top_{(L_2)}$.

We say that f preserves bottom if and only if:

(Def.8) $f(\bot_{(L_1)}) = \bot_{(L_2)}$.

We say that f preserves complement if and only if:

(Def.9) $f({a_1}^{\mathrm{c}}) = f(a_1)^{\mathrm{c}}$.

Let us consider L. A non empty subset of the carrier of L is said to be a closed subset of L if:

(Def.10) If $p \in$ it and $q \in$ it, then $p \sqcap q \in$ it and $p \sqcup q \in$ it.

Next we state two propositions:

(10) The carrier of L is a closed subset of L.

(11) Every filter of L is a closed subset of L.

Let L be a lattice. The functor id_L yields a function from the carrier of L into the carrier of L and is defined as follows:

(Def.11) $\mathrm{id}_L = \mathrm{id}_{(\text{the carrier of } L)}$.

Next we state two propositions:

(12) For every element b of the carrier of L holds $\mathrm{id}_L(b) = b$.

(13) For every function f from the carrier of L into the carrier of L holds $f \cdot \mathrm{id}_L = f$ and $\mathrm{id}_L \cdot f = f$.

In the sequel B denotes a finite subset of the carrier of L.

Let us consider L, B. The functor $\bigsqcup^{\mathrm{f}}_B$ yields an element of the carrier of L and is defined by:

(Def.12) $\bigsqcup^{\mathrm{f}}_B = \bigsqcup^{\mathrm{f}}_B(\mathrm{id}_L)$.

The functor $\bigsqcap^{\mathrm{f}}_B$ yielding an element of the carrier of L is defined by:

(Def.13) $\bigsqcap^{\mathrm{f}}_B = \bigsqcap^{\mathrm{f}}_B(\mathrm{id}_L)$.

The following propositions are true:

(14) $\bigsqcap^{\mathrm{f}}_B =$ (the meet operation of L)-$\sum_B \mathrm{id}_L$.

(15) $\bigsqcup^{\mathrm{f}}_B =$ (the join operation of L)-$\sum_B \mathrm{id}_L$.

(16) $\bigsqcup^{\mathrm{f}}_{\{p\}} = p$.

(17) $\bigsqcap^{\mathrm{f}}_{\{p\}} = p$.

3. Distributive Lattices

In the sequel D_1 denotes a distributive lattice and f denotes a homomorphism from D_1 to L_2.

One can prove the following proposition

(18) If f is epimorphism, then L_2 is distributive.

4. Lower-bounded Lattices

We adopt the following rules: ℓ_1 is a lower-bounded lattice, B, B_1, B_2 are finite subsets of the carrier of ℓ_1, and b is an element of the carrier of ℓ_1.

Next we state the proposition

(19) Let f be a homomorphism from ℓ_1 to L_2. If f is epimorphism, then L_2 is lower-bounded and f preserves bottom.

In the sequel f will be a unary operation on the carrier of ℓ_1.

We now state several propositions:

(20) $\bigsqcup^{\mathrm{f}}_{B \cup \{b\}} f = \bigsqcup^{\mathrm{f}}_B f \sqcup f(b)$.

(21) $\bigsqcup^{\mathrm{f}}_{B \cup \{b\}} = \bigsqcup^{\mathrm{f}}_B \sqcup b$.

(22) $\bigsqcup^{\mathrm{f}}_{(B_1)} \sqcup \bigsqcup^{\mathrm{f}}_{(B_2)} = \bigsqcup^{\mathrm{f}}_{B_1 \cup B_2}$.

(23) $\bigsqcup^{\mathrm{f}}_{\emptyset_{\text{the carrier of } \ell_1}} = \bot_{(\ell_1)}$.

(24) For every closed subset A of ℓ_1 such that $\bot_{(\ell_1)} \in A$ and for every B such that $B \subseteq A$ holds $\bigsqcup^{\mathrm{f}}_{B} \in A$.

5. Upper-bounded Lattices

We adopt the following rules: ℓ_2 will denote an upper-bounded lattice, B, B_1, B_2 will denote finite subsets of the carrier of ℓ_2, and b will denote an element of the carrier of ℓ_2.

One can prove the following two propositions:

(25) For every homomorphism f from ℓ_2 to L_2 such that f is epimorphism holds L_2 is upper-bounded and f preserves top.

(26) $\bigsqcap^{\mathrm{f}}_{\emptyset_{\text{the carrier of } \ell_2}} = \top_{(\ell_2)}$.

In the sequel f, g will be unary operations on the carrier of ℓ_2.

The following propositions are true:

(27) $\bigsqcap^{\mathrm{f}}_{B \cup \{b\}} f = \bigsqcap^{\mathrm{f}}_{B} f \sqcap f(b)$.

(28) $\bigsqcap^{\mathrm{f}}_{B \cup \{b\}} = \bigsqcap^{\mathrm{f}}_{B} \sqcap b$.

(29) $\bigsqcap^{\mathrm{f}}_{f \circ B} g = \bigsqcap^{\mathrm{f}}_{B} (g \cdot f)$.

(30) $\bigsqcap^{\mathrm{f}}_{(B_1)} \sqcap \bigsqcap^{\mathrm{f}}_{(B_2)} = \bigsqcap^{\mathrm{f}}_{B_1 \cup B_2}$.

(31) For every closed subset F of ℓ_2 such that $\top_{(\ell_2)} \in F$ and for every B such that $B \subseteq F$ holds $\bigsqcap^{\mathrm{f}}_{B} \in F$.

6. Distributive Upper-bounded Lattices

In the sequel D_1 will be a distributive upper-bounded lattice, B will be a finite subset of the carrier of D_1, and p will be an element of the carrier of D_1.

Next we state the proposition

(32) $\bigsqcap^{\mathrm{f}}_{B} \sqcup p = \bigsqcap^{\mathrm{f}}_{((\text{the join operation of } D_1)^{\circ}(\mathrm{id}_{(D_1)}, p))^{\circ} B}$.

7. Implicative Lattices

For simplicity we adopt the following rules: C_1 denotes a complemented lattice, I_1 denotes an implicative lattice, f denotes a homomorphism from I_1 to C_1, and i, j, k denote elements of the carrier of I_1.

The following propositions are true:

(33) $f(i) \sqcap f(i \Rightarrow j) \sqsubseteq f(j)$.

(34) If f is monomorphism, then if $f(i) \sqcap f(k) \sqsubseteq f(j)$, then $f(k) \sqsubseteq f(i \Rightarrow j)$.

(35) If f is isomorphism, then C_1 is implicative and f preserves implication.

8. Boolean Lattices

For simplicity we adopt the following rules: B_3 will be a Boolean lattice, f will be a homomorphism from B_3 to C_1, A will be a non empty subset of the carrier of B_3, a, b, c, p, q will be elements of the carrier of B_3, and B, B_0 will be finite subsets of the carrier of B_3.

One can prove the following propositions:

(36) $(\top_{(B_3)})^{\rm c} = \bot_{(B_3)}$.

(37) $(\bot_{(B_3)})^{\rm c} = \top_{(B_3)}$.

(38) If f is epimorphism, then C_1 is Boolean and f preserves complement.

Let us consider B_3. A non empty subset of the carrier of B_3 is called a field of subsets of B_3 if:

(Def.14) If $a \in$ it and $b \in$ it, then $a \sqcap b \in$ it and $a^{\rm c} \in$ it.

In the sequel F will denote a field of subsets of B_3.

Next we state four propositions:

(39) If $a \in F$ and $b \in F$, then $a \sqcup b \in F$.

(40) If $a \in F$ and $b \in F$, then $a \Rightarrow b \in F$.

(41) The carrier of B_3 is a field of subsets of B_3.

(42) F is a closed subset of B_3.

Let us consider B_3, A. The field by A yielding a field of subsets of B_3 is defined as follows:

(Def.15) $A \subseteq$ the field by A and for every F such that $A \subseteq F$ holds the field by $A \subseteq F$.

Let us consider B_3, A. The functor SetImp(A) yielding a non empty subset of the carrier of B_3 is defined by:

(Def.16) $\mathrm{SetImp}(A) = \{a \Rightarrow b : a \in A \ \wedge \ b \in A\}$.

The following two propositions are true:

(43) $x \in \mathrm{SetImp}(A)$ iff there exist p, q such that $x = p \Rightarrow q$ and $p \in A$ and $q \in A$.

(44) $c \in \mathrm{SetImp}(A)$ iff there exist p, q such that $c = p^{\rm c} \sqcup q$ and $p \in A$ and $q \in A$.

Let us consider B_3. The functor comp B_3 yielding a function from the carrier of B_3 into the carrier of B_3 is defined by:

(Def.17) $(\mathrm{comp}\, B_3)(a) = a^{\rm c}$.

We now state several propositions:

(45) $\bigsqcup^{\rm f}_{B \cup \{b\}} \mathrm{comp}\, B_3 = \bigsqcup^{\rm f}_B \mathrm{comp}\, B_3 \sqcup b^{\rm c}$.

(46) $(\bigsqcup^{\rm f}_B)^{\rm c} = \bigsqcap^{\rm f}_B \mathrm{comp}\, B_3$.

(47) $\bigsqcap^{\rm f}_{B \cup \{b\}} \mathrm{comp}\, B_3 = \bigsqcap^{\rm f}_B \mathrm{comp}\, B_3 \sqcap b^{\rm c}$.

(48) $(\bigsqcap^{\rm f}_B)^{\rm c} = \bigsqcup^{\rm f}_B \mathrm{comp}\, B_3$.

(49) Let A_1 be a closed subset of B_3. Suppose $\bot_{(B_3)} \in A_1$ and $\top_{(B_3)} \in A_1$. Given B. If $B \subseteq \mathrm{SetImp}(A_1)$, then there exists B_0 such that $B_0 \subseteq \mathrm{SetImp}(A_1)$ and $\bigsqcup^{\mathrm{f}}_{B} \mathrm{comp}\, B_3 = \bigsqcap^{\mathrm{f}}_{(B_0)}$.

(50) For every closed subset A_1 of B_3 such that $\bot_{(B_3)} \in A_1$ and $\top_{(B_3)} \in A_1$ holds $\{\bigsqcap^{\mathrm{f}}_{B} : B \subseteq \mathrm{SetImp}(A_1)\}$ = the field by A_1.

References

[1] Grzegorz Bancerek. Filters - part I. *Formalized Mathematics*, 1(**5**):813–819, 1990.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[5] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[6] Andrzej Trybulec. Finite join and finite meet and dual lattices. *Formalized Mathematics*, 1(**5**):983–988, 1990.
[7] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.
[8] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.
[9] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[10] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

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