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# Introduction to Theory of Rearrangement [1]

Yuji Sakai
Shinshu University
Nagano

Jarosław Kotowicz
Warsaw University
Białystok

**Summary.** An introduction to the rearrangement theory for finite functions (e.g. with the finite domain and codomain). The notion of generators and cogenerators of finite sets (equivalent to the order in the language of finite sequences) has been defined. The notion of rearrangement for a function into finite set is presented. Some basic properties of these notions have been proved.

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The terminology and notation used here are introduced in the following articles: [15], [5], [3], [1], [8], [10], [2], [16], [6], [4], [7], [12], [13], [9], [11], and [14].

Let $D$ be a non empty set, let $F$ be a partial function from $D$ to $\mathbb{R}$, and let $r$ be a real number. Then $r\,F$ is an element of $D \dot{\rightarrow} \mathbb{R}$.

A finite sequence has cardinality by index if:

(Def.1) For every $n$ such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ holds $\operatorname{card} \mathrm{it}(n) = n$.

A finite sequence is ascending if:

(Def.2) For every $n$ such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it} - 1$ holds $\mathrm{it}(n) \subseteq \mathrm{it}(n+1)$.

Let $X$ be a set. A finite sequence of elements of $X$ has length by cardinality if:

(Def.3) $\operatorname{len} \mathrm{it} = \operatorname{card} \bigcup X$.

Let $D$ be a non empty finite set. Note that there exists a finite sequence of elements of $2^D$ which is ascending and has cardinality by index and length by cardinality.

Let $D$ be a non empty finite set. A rearrangement generator of $D$ is an ascending finite sequence of elements of $2^D$ with cardinality by index and length by cardinality.

One can prove the following propositions:

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(1) For every finite sequence $a$ of elements of $2^D$ holds $a$ has length by cardinality iff $\operatorname{len} a = \operatorname{card} D$.

(2) Let $a$ be a finite sequence. Then $a$ is ascending if and only if for all $n$, $m$ such that $n \leq m$ and $n \in \operatorname{dom} a$ and $m \in \operatorname{dom} a$ holds $a(n) \subseteq a(m)$.

(3) For every finite sequence $a$ of elements of $2^D$ with cardinality by index and length by cardinality holds $a(\operatorname{len} a) = D$.

(4) For every finite sequence $a$ of elements of $2^D$ with length by cardinality holds $\operatorname{len} a \neq 0$.

(5) Let $a$ be an ascending finite sequence of elements of $2^D$ with cardinality by index and given $n$, $m$. If $n \in \operatorname{dom} a$ and $m \in \operatorname{dom} a$ and $n \neq m$, then $a(n) \neq a(m)$.

(6) Let $a$ be an ascending finite sequence of elements of $2^D$ with cardinality by index and given $n$. If $1 \leq n$ and $n \leq \operatorname{len} a - 1$, then $a(n) \neq a(n+1)$.

(7) For every finite sequence $a$ of elements of $2^D$ with cardinality by index such that $n \in \operatorname{dom} a$ holds $a(n) \neq \emptyset$.

(8) Let $a$ be a finite sequence of elements of $2^D$ with cardinality by index. If $1 \leq n$ and $n \leq \operatorname{len} a - 1$, then $a(n+1) \setminus a(n) \neq \emptyset$.

(9) Let $a$ be a finite sequence of elements of $2^D$ with cardinality by index and length by cardinality. Then there exists an element $d$ of $D$ such that $a(1) = \{d\}$.

(10) Let $a$ be an ascending finite sequence of elements of $2^D$ with cardinality by index. Suppose $1 \leq n$ and $n \leq \operatorname{len} a - 1$. Then there exists an element $d$ of $D$ such that $a(n+1) \setminus a(n) = \{d\}$ and $a(n+1) = a(n) \cup \{d\}$ and $a(n+1) \setminus \{d\} = a(n)$.

Let $D$ be a non empty finite set and let $A$ be a rearrangement generator of $D$. The functor co-Gen$(A)$ yielding a rearrangement generator of $D$ is defined by:

(Def.4) For every $m$ such that $1 \leq m$ and $m \leq \operatorname{len} \text{co-Gen}(A) - 1$ holds $(\text{co-Gen}(A))(m) = D \setminus A(\operatorname{len} A - m)$.

One can prove the following two propositions:

(11) For every rearrangement generator $A$ of $D$ holds co-Gen(co-Gen$(A)$) = $A$.

(12) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{len} \mathrm{MIM}(\mathrm{FinS}(F, D)) = \operatorname{len} \mathrm{CHI}(A, C)$.

Let $D$, $C$ be non empty finite set, let $A$ be a rearrangement generator of $C$, and let $F$ be a partial function from $D$ to $\mathbb{R}$. The functor $F_A^\wedge$ yields a partial function from $C$ to $\mathbb{R}$ and is defined by:

(Def.5) $F_A^\wedge = \sum(\mathrm{MIM}(\mathrm{FinS}(F, D))\, \mathrm{CHI}(A, C))$.

The functor $F_A^\vee$ yields a partial function from $C$ to $\mathbb{R}$ and is defined as follows:

(Def.6) $F_A^\vee = \sum(\mathrm{MIM}(\mathrm{FinS}(F, D))\, \mathrm{CHI}(\text{co-Gen}(A), C))$.

Next we state a number of propositions:

(13) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{dom} F^{\wedge}_{A} = C$.

(14) Let $c$ be an element of $C$, and let $F$ be a partial function from $D$ to $\mathbb{R}$, and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then

(i) if $c \in A(1)$, then $(\mathrm{MIM}(\mathrm{FinS}(F, D))\,\mathrm{CHI}(A, C))\#c = \mathrm{MIM}(\mathrm{FinS}(F, D))$, and

(ii) for every $n$ such that $1 \leq n$ and $n < \operatorname{len} A$ and $c \in A(n+1) \setminus A(n)$ holds $(\mathrm{MIM}(\mathrm{FinS}(F, D))\,\mathrm{CHI}(A, C))\#c = (n \longmapsto (0$ **qua** real number$))$ $^\frown \mathrm{MIM}((\mathrm{FinS}(F, D))_{\restriction n})$.

(15) Let $c$ be an element of $C$, and let $F$ be a partial function from $D$ to $\mathbb{R}$, and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then if $c \in A(1)$, then $(F^{\wedge}_{A})(c) = (\mathrm{FinS}(F, D))(1)$ and for every $n$ such that $1 \leq n$ and $n < \operatorname{len} A$ and $c \in A(n+1) \setminus A(n)$ holds $(F^{\wedge}_{A})(c) = (\mathrm{FinS}(F, D))(n+1)$.

(16) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{rng} F^{\wedge}_{A} = \operatorname{rng} \mathrm{FinS}(F, D)$.

(17) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $F^{\wedge}_{A}$ and $\mathrm{FinS}(F, D)$ are fiberwise equipotent.

(18) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\mathrm{FinS}(F^{\wedge}_{A}, C) = \mathrm{FinS}(F, D)$.

(19) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\sum_{\kappa=0}^{C} F^{\wedge}_{A}(\kappa) = \sum_{\kappa=0}^{D} F(\kappa)$.

(20) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\mathrm{FinS}((F^{\wedge}_{A}) - r, C) = \mathrm{FinS}(F - r, D)$ and $\sum_{\kappa=0}^{C}((F^{\wedge}_{A}) - r)(\kappa) = \sum_{\kappa=0}^{D}(F - r)(\kappa)$.

(21) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{dom} F^{\vee}_{A} = C$.

(22) Let $c$ be an element of $C$, and let $F$ be a partial function from $D$ to $\mathbb{R}$, and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then if $c \in (\text{co-Gen}(A))(1)$, then $(F^{\vee}_{A})(c) = (\mathrm{FinS}(F, D))(1)$ and for every $n$ such that $1 \leq n$ and $n < \operatorname{len} \text{co-Gen}(A)$ and $c \in (\text{co-Gen}(A))(n+1) \setminus (\text{co-Gen}(A))(n)$ holds $(F^{\vee}_{A})(c) = (\mathrm{FinS}(F, D))(n+1)$.

(23) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\operatorname{rng} F^{\vee}_{A} = \operatorname{rng} \mathrm{FinS}(F, D)$.

(24) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $F^{\vee}_{A}$ and

$\mathrm{FinS}(F, D)$ are fiberwise equipotent.

(25) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\mathrm{FinS}(F_A^{\vee}, C) = \mathrm{FinS}(F, D)$.

(26) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\sum_{\kappa=0}^{C} F_A^{\vee}(\kappa) = \sum_{\kappa=0}^{D} F(\kappa)$.

(27) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} C = \operatorname{card} D$, then $\mathrm{FinS}((F_A^{\vee}) - r, C) = \mathrm{FinS}(F - r, D)$ and $\sum_{\kappa=0}^{C}((F_A^{\vee}) - r)(\kappa) = \sum_{\kappa=0}^{D}(F - r)(\kappa)$.

(28) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $F_A^{\vee}$ and $F_A^{\wedge}$ are fiberwise equipotent and $\mathrm{FinS}(F_A^{\vee}, C) = \mathrm{FinS}(F_A^{\wedge}, C)$ and $\sum_{\kappa=0}^{C} F_A^{\vee}(\kappa) = \sum_{\kappa=0}^{C} F_A^{\wedge}(\kappa)$.

(29) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_+((F_A^{\wedge}) - r)$ and $\max_+(F - r)$ are fiberwise equipotent and $\mathrm{FinS}(\max_+((F_A^{\wedge}) - r), C) = \mathrm{FinS}(\max_+(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_+((F_A^{\wedge}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_+(F - r)(\kappa)$.

(30) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_-((F_A^{\wedge}) - r)$ and $\max_-(F - r)$ are fiberwise equipotent and $\mathrm{FinS}(\max_-((F_A^{\wedge}) - r), C) = \mathrm{FinS}(\max_-(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_-((F_A^{\wedge}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_-(F - r)(\kappa)$.

(31) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$, then $\operatorname{len} \mathrm{FinS}(F_A^{\wedge}, C) = \operatorname{card} C$ and $1 \leq \operatorname{len} \mathrm{FinS}(F_A^{\wedge}, C)$.

(32) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$ and $n \in \operatorname{dom} A$, then $\mathrm{FinS}(F_A^{\wedge}, C) \restriction n = \mathrm{FinS}(F_A^{\wedge}, A(n))$.

(33) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$, then $(F - r)_A^{\wedge} = (F_A^{\wedge}) - r$.

(34) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_+((F_A^{\vee}) - r)$ and $\max_+(F - r)$ are fiberwise equipotent and $\mathrm{FinS}(\max_+((F_A^{\vee}) - r), C) = \mathrm{FinS}(\max_+(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_+((F_A^{\vee}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_+(F - r)(\kappa)$.

(35) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} C = \operatorname{card} D$. Then $\max_-((F_A^{\vee}) - r)$ and $\max_-(F - r)$ are fiberwise equipotent and $\mathrm{FinS}(\max_-((F_A^{\vee}) - r), C) = \mathrm{FinS}(\max_-(F - r), D)$ and $\sum_{\kappa=0}^{C} \max_-((F_A^{\vee}) - r)(\kappa) = \sum_{\kappa=0}^{D} \max_-(F - r)(\kappa)$.

(36) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$, then $\operatorname{len} \operatorname{FinS}(F_A^{\vee}, C) = \operatorname{card} C$ and $1 \leq \operatorname{len} \operatorname{FinS}(F_A^{\vee}, C)$.

(37) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$ and $n \in \operatorname{dom} A$, then $\operatorname{FinS}(F_A^{\vee}, C) \restriction n = \operatorname{FinS}(F_A^{\vee}, (\text{co-Gen}(A))(n))$.

(38) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. If $F$ is total and $\operatorname{card} D = \operatorname{card} C$, then $(F - r)_A^{\vee} = (F_A^{\vee}) - r$.

(39) Let $F$ be a partial function from $D$ to $\mathbb{R}$ and let $A$ be a rearrangement generator of $C$. Suppose $F$ is total and $\operatorname{card} D = \operatorname{card} C$. Then $F_A^{\wedge}$ and $F$ are fiberwise equipotent and $F_A^{\vee}$ and $F$ are fiberwise equipotent and $\operatorname{rng} F_A^{\wedge} = \operatorname{rng} F$ and $\operatorname{rng} F_A^{\vee} = \operatorname{rng} F$.

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