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Homomorphisms of Many Sorted Algebras

Małgorzata Korolkiewicz
Warsaw University
Białystok

Summary. The aim of this article is to present the definition and some properties of homomorphisms of many sorted algebras. Some auxiliary properties of many sorted functions also have been shown.

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The notation and terminology used in this paper have been introduced in the following articles: [10], [12], [13], [5], [6], [2], [4], [1], [11], [9], [7], [8], and [3].

1. Preliminaries

For simplicity we follow the rules: S is a non void non empty many sorted signature, U_1, U_2, U_3 are non-empty algebras over S, o is an operation symbol of S, and n is a natural number.

Let I be a non empty set, let A, B be non-empty many sorted sets of I, let F be a many sorted function from A into B, and let i be an element of I. Then $F(i)$ is a function from $A(i)$ into $B(i)$.

Let us consider S, U_1, U_2. A many sorted function from U_1 into U_2 is a many sorted function from the sorts of U_1 into the sorts of U_2.

Let I be a set and let A be a many sorted set of I. The functor id_A yields a many sorted function from A into A and is defined as follows:

(Def.1) For arbitrary i such that $i \in I$ holds $\mathrm{id}_A(i) = \mathrm{id}_{A(i)}$.

A function is "1-1" if:

(Def.2) For arbitrary i and for every function f such that $i \in \mathrm{dom}\,\mathrm{it}$ and $\mathrm{it}(i) = f$ holds f is one-to-one.

Let I be a set. Observe that there exists a many sorted function of I which is "1-1".

We now state the proposition

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(1) Let I be a set and let F be a many sorted function of I. Then F is "1-1" if and only if for arbitrary i and for every function f such that $i \in I$ and $F(i) = f$ holds f is one-to-one.

Let I be a set and let A, B be many sorted sets of I. A many sorted function from A into B is "onto" if:

(Def.3) For arbitrary i and for every function f from $A(i)$ into $B(i)$ such that $i \in I$ and $\mathrm{it}(i) = f$ holds $\mathrm{rng}\, f = B(i)$.

Let F, G be function yielding functions. The functor $G \circ F$ yielding a function yielding function is defined by the conditions (Def.4).

(Def.4) (i) $\mathrm{dom}(G \circ F) = \mathrm{dom}\, F \cap \mathrm{dom}\, G$, and

(ii) for arbitrary i and for every function f and for every function g such that $i \in \mathrm{dom}(G \circ F)$ and $f = F(i)$ and $g = G(i)$ holds $(G \circ F)(i) = g \cdot f$.

We now state the proposition

(2) Let I be a set, and let A be a many sorted set of I, and let B, C be non-empty many sorted sets of I, and let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Then

(i) $\mathrm{dom}(G \circ F) = I$, and

(ii) for arbitrary i and for every function f from $A(i)$ into $B(i)$ and for every function g from $B(i)$ into $C(i)$ such that $i \in I$ and $f = F(i)$ and $g = G(i)$ holds $(G \circ F)(i) = g \cdot f$.

Let I be a set, let A be a many sorted set of I, let B, C be non-empty many sorted sets of I, let F be a many sorted function from A into B, and let G be a many sorted function from B into C. Then $G \circ F$ is a many sorted function from A into C.

Next we state two propositions:

(3) Let I be a set, and let A, B be non-empty many sorted sets of I, and let F be a many sorted function from A into B. Then $F \circ \mathrm{id}_A = F$.

(4) Let I be a set, and let A be a many sorted set of I, and let B be a non-empty many sorted set of I, and let F be a many sorted function from A into B. Then $\mathrm{id}_B \circ F = F$.

Let I be a set, let A, B be non-empty many sorted sets of I, and let F be a many sorted function from A into B. Let us assume that F is "1-1" and "onto". The functor F^{-1} yielding a many sorted function from B into A is defined as follows:

(Def.5) For arbitrary i and for every function f from $A(i)$ into $B(i)$ such that $i \in I$ and $f = F(i)$ holds $F^{-1}(i) = f^{-1}$.

We now state the proposition

(5) Let I be a set, and let A, B be non-empty many sorted sets of I, and let H be a many sorted function from A into B, and let H_1 be a many sorted function from B into A. If H is "1-1" and "onto" and $H_1 = H^{-1}$, then $H \circ H_1 = \mathrm{id}_B$ and $H_1 \circ H = \mathrm{id}_A$.

Let I be a set, let A be a many sorted set of I, and let F be a many sorted function of I. The functor $F \circ A$ yields a many sorted set of I and is defined as follows:

(Def.6) For arbitrary i and for every function f such that $i \in I$ and $f = F(i)$ holds $(F \circ A)(i) = f^\circ A(i)$.

Let us consider S, U_1, o. Observe that every element of $\mathrm{Args}(o, U_1)$ is function-like and relation-like.

2. Homomorphisms of Many Sorted Algebras

One can prove the following proposition

(6) Let x be an element of $\mathrm{Args}(o, U_1)$. Then $\mathrm{dom}\, x = \mathrm{dom}\, \mathrm{Arity}(o)$ and for arbitrary y such that $y \in \mathrm{dom}((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))$ holds $x(y) \in ((\text{the sorts of } U_1) \cdot \mathrm{Arity}(o))(y)$.

Let us consider S, U_1, U_2, o, let F be a many sorted function from U_1 into U_2, and let x be an element of $\mathrm{Args}(o, U_1)$. The functor $F\#x$ yielding an element of $\mathrm{Args}(o, U_2)$ is defined by:

(Def.7) For every n such that $n \in \mathrm{dom}\, x$ holds $(F\#x)(n) = F(\pi_n \mathrm{Arity}(o))(x(n))$.

The following two propositions are true:

(7) For all S, o, U_1 and for every element x of $\mathrm{Args}(o, U_1)$ holds $x = \mathrm{id}_{(\text{the sorts of } U_1)}\#x$.

(8) Let H_1 be a many sorted function from U_1 into U_2, and let H_2 be a many sorted function from U_2 into U_3, and let x be an element of $\mathrm{Args}(o, U_1)$. Then $(H_2 \circ H_1)\#x = H_2\#(H_1\#x)$.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is a homomorphism of U_1 into U_2 if and only if:

(Def.8) For every operation symbol o of S and for every element x of $\mathrm{Args}(o, U_1)$ holds $F(\text{the result sort of } o)((\mathrm{Den}(o, U_1))(x)) = (\mathrm{Den}(o, U_2))(F\#x)$.

Next we state two propositions:

(9) $\mathrm{id}_{(\text{the sorts of } U_1)}$ is a homomorphism of U_1 into U_1.

(10) Let H_1 be a many sorted function from U_1 into U_2 and let H_2 be a many sorted function from U_2 into U_3. Suppose H_1 is a homomorphism of U_1 into U_2 and H_2 is a homomorphism of U_2 into U_3. Then $H_2 \circ H_1$ is a homomorphism of U_1 into U_3.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is an epimorphism of U_1 onto U_2 if and only if:

(Def.9) F is a homomorphism of U_1 into U_2 and "onto".

One can prove the following proposition

(11) Let F be a many sorted function from U_1 into U_2 and let G be a many sorted function from U_2 into U_3. Suppose F is an epimorphism of U_1 onto U_2 and G is an epimorphism of U_2 onto U_3. Then $G \circ F$ is an epimorphism of U_1 onto U_3.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is a monomorphism of U_1 into U_2 if and only if:

(Def.10) F is a homomorphism of U_1 into U_2 and "1-1".

The following proposition is true

(12) Let F be a many sorted function from U_1 into U_2 and let G be a many sorted function from U_2 into U_3. Suppose F is a monomorphism of U_1 into U_2 and G is a monomorphism of U_2 into U_3. Then $G \circ F$ is a monomorphism of U_1 into U_3.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. We say that F is an isomorphism of U_1 and U_2 if and only if:

(Def.11) F is an epimorphism of U_1 onto U_2 and a monomorphism of U_1 into U_2.

The following propositions are true:

(13) Let F be a many sorted function from U_1 into U_2. Then F is an isomorphism of U_1 and U_2 if and only if F is a homomorphism of U_1 into U_2 "onto" and "1-1".

(14) Let H be a many sorted function from U_1 into U_2 and let H_1 be a many sorted function from U_2 into U_1. Suppose H is an isomorphism of U_1 and U_2 and $H_1 = H^{-1}$. Then H_1 is an isomorphism of U_2 and U_1.

(15) Let H be a many sorted function from U_1 into U_2 and let H_1 be a many sorted function from U_2 into U_3. Suppose H is an isomorphism of U_1 and U_2 and H_1 is an isomorphism of U_2 and U_3. Then $H_1 \circ H$ is an isomorphism of U_1 and U_3.

Let us consider S, U_1, U_2. We say that U_1 and U_2 are isomorphic if and only if:

(Def.12) There exists many sorted function from U_1 into U_2 which is an isomorphism of U_1 and U_2.

Next we state three propositions:

(16) U_1 and U_1 are isomorphic.

(17) If U_1 and U_2 are isomorphic, then U_2 and U_1 are isomorphic.

(18) If U_1 and U_2 are isomorphic and U_2 and U_3 are isomorphic, then U_1 and U_3 are isomorphic.

Let us consider S, U_1, U_2 and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2. The functor $\operatorname{Im} F$ yields a strict non-empty subalgebra of U_2 and is defined as follows:

(Def.13) The sorts of $\operatorname{Im} F = F^{\circ}$ (the sorts of U_1).

We now state several propositions:

(19) Let U_2 be a strict non-empty algebra over S and let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then F is an epimorphism of U_1 onto U_2 if and only if $\mathrm{Im}\,F = U_2$.

(20) Let F be a many sorted function from U_1 into U_2 and let G be a many sorted function from U_1 into $\mathrm{Im}\,F$. Suppose $F = G$ and F is a homomorphism of U_1 into U_2. Then G is an epimorphism of U_1 onto $\mathrm{Im}\,F$.

(21) Let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function G from U_1 into $\mathrm{Im}\,F$ such that $F = G$ and G is an epimorphism of U_1 onto $\mathrm{Im}\,F$.

(22) Let U_2 be a strict non-empty subalgebra of U_1 and let G be a many sorted function from U_2 into U_1. If $G = \mathrm{id}_{(\text{the sorts of } U_2)}$, then G is a monomorphism of U_2 into U_1.

(23) Let F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2. Then there exists a many sorted function F_1 from U_1 into $\mathrm{Im}\,F$ and there exists a many sorted function F_2 from $\mathrm{Im}\,F$ into U_2 such that F_1 is an epimorphism of U_1 onto $\mathrm{Im}\,F$ and F_2 is a monomorphism of $\mathrm{Im}\,F$ into U_2 and $F = F_2 \circ F_1$.

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