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Certain Facts about Families of Subsets of Many Sorted Sets

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The terminology and notation used in this paper are introduced in the following papers: [22], [23], [6], [19], [16], [24], [3], [4], [2], [7], [18], [5], [21], [20], [1], [12], [13], [14], [10], [15], [9], [17], [8], and [11].

1. Preliminaries

For simplicity we follow the rules: I, G, H will denote sets, i will be arbitrary, A, B, M will denote many sorted sets indexed by I, s_1, s_2, s_3 will denote families of subsets of I, v, w will denote subsets of I, and F will denote a many sorted function of I.

The scheme *MSFExFunc* deals with a set $\mathcal{A}$, a many sorted set $\mathcal{B}$ indexed by $\mathcal{A}$, a many sorted set $\mathcal{C}$ indexed by $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exists a many sorted function F from $\mathcal{B}$ into $\mathcal{C}$ such that for arbitrary i if $i \in \mathcal{A}$, then there exists a function f from $\mathcal{B}(i)$ into $\mathcal{C}(i)$ such that $f = F(i)$ and for arbitrary x such that $x \in \mathcal{B}(i)$ holds $\mathcal{P}[f(x), x, i]$

provided the following condition is satisfied:

- Let i be arbitrary. Suppose $i \in \mathcal{A}$. Let x be arbitrary. If $x \in \mathcal{B}(i)$, then there exists arbitrary y such that $y \in \mathcal{C}(i)$ and $\mathcal{P}[y, x, i]$.

We now state a number of propositions:

(1) If $s_1 \neq \emptyset$, then $\mathrm{Intersect}(s_1) \subseteq \bigcup s_1$.

(2) If $G \in s_1$, then $\mathrm{Intersect}(s_1) \subseteq G$.

(3) If $\emptyset \in s_1$, then $\mathrm{Intersect}(s_1) = \emptyset$.

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(4) For every subset Z of I such that for arbitrary Z_1 such that $Z_1 \in s_1$ holds $Z \subseteq Z_1$ holds $Z \subseteq \mathrm{Intersect}(s_1)$.

(5) If $s_1 \neq \emptyset$ and for every set Z_1 such that $Z_1 \in s_1$ holds $G \subseteq Z_1$, then $G \subseteq \mathrm{Intersect}(s_1)$.

(6) If $G \in s_1$ and $G \subseteq H$, then $\mathrm{Intersect}(s_1) \subseteq H$.

(7) If $G \in s_1$ and $G \cap H = \emptyset$, then $\mathrm{Intersect}(s_1) \cap H = \emptyset$.

(8) If $s_3 = s_1 \cup s_2$, then $\mathrm{Intersect}(s_3) = \mathrm{Intersect}(s_1) \cap \mathrm{Intersect}(s_2)$.

(9) If $s_1 = \{v\}$, then $\mathrm{Intersect}(s_1) = v$.

(10) If $s_1 = \{v, w\}$, then $\mathrm{Intersect}(s_1) = v \cap w$.

(11) If $A \in B$, then A is an element of B.

(12) For every non-empty many sorted set B indexed by I such that A is an element of B holds $A \in B$.

(13) For every function f such that $i \in I$ and $f = F(i)$ holds $(\mathrm{rng}_\kappa F(\kappa))(i) = \mathrm{rng}\, f$.

(14) For every function f such that $i \in I$ and $f = F(i)$ holds $(\mathrm{dom}_\kappa F(\kappa))(i) = \mathrm{dom}\, f$.

(15) For all many sorted functions F, G of I holds $G \circ F$ is a many sorted function of I.

(16) Let A be a non-empty many sorted set indexed by I and let F be a many sorted function from A into $\emptyset_I$. Then $F = \emptyset_I$.

(17) If A is transformable to B and F is a many sorted function from A into B, then $\mathrm{dom}_\kappa F(\kappa) = A$ and $\mathrm{rng}_\kappa F(\kappa) \subseteq B$.

2. Finite Many Sorted Sets

Let us consider I. Note that every many sorted set indexed by I which is empty yielding is also locally-finite.

Let us consider I. Note that $\emptyset_I$ is empty yielding and locally-finite.

Let us consider I, A. Note that there exists a many sorted subset of A which is empty yielding and locally-finite.

Next we state the proposition

(18) If $A \subseteq B$ and B is locally-finite, then A is locally-finite.

Let us consider I and let A be a locally-finite many sorted set indexed by I. One can check that every many sorted subset of A is locally-finite.

Let us consider I and let A, B be locally-finite many sorted sets indexed by I. Note that $A \cup B$ is locally-finite.

Let us consider I, A and let B be a locally-finite many sorted set indexed by I. Note that $A \cap B$ is locally-finite.

Let us consider I, B and let A be a locally-finite many sorted set indexed by I. Observe that $A \cap B$ is locally-finite.

Let us consider I, B and let A be a locally-finite many sorted set indexed by I. Note that $A \setminus B$ is locally-finite.

Let us consider I, F and let A be a locally-finite many sorted set indexed by I. Observe that $F \circ A$ is locally-finite.

Let us consider I and let A, B be locally-finite many sorted sets indexed by I. Observe that $[\![A, B]\!]$ is locally-finite.

The following propositions are true:

(19) If B is non-empty and $[\![A, B]\!]$ is locally-finite, then A is locally-finite.

(20) If A is non-empty and $[\![A, B]\!]$ is locally-finite, then B is locally-finite.

(21) A is locally-finite iff 2^A is locally-finite.

Let us consider I and let M be a locally-finite many sorted set indexed by I. Observe that 2^M is locally-finite.

The following propositions are true:

(22) Let A be a non-empty many sorted set indexed by I. Suppose A is locally-finite and for every many sorted set M indexed by I such that $M \in A$ holds M is locally-finite. Then $\bigcup A$ is locally-finite.

(23) If $\bigcup A$ is locally-finite, then A is locally-finite and for every M such that $M \in A$ holds M is locally-finite.

(24) If $\mathrm{dom}_\kappa\, F(\kappa)$ is locally-finite, then $\mathrm{rng}_\kappa\, F(\kappa)$ is locally-finite.

(25) Suppose $A \subseteq \mathrm{rng}_\kappa\, F(\kappa)$ and for arbitrary i and for every function f such that $i \in I$ and $f = F(i)$ holds $f^{-1}\, A(i)$ is finite. Then A is locally-finite.

Let us consider I and let A, B be locally-finite many sorted sets indexed by I. Observe that $\mathrm{MSFuncs}(A, B)$ is locally-finite.

Let us consider I and let A, B be locally-finite many sorted sets indexed by I. Note that $A \dot{-} B$ is locally-finite.

In the sequel X, Y, Z denote many sorted sets indexed by I.

One can prove the following propositions:

(26) Suppose X is locally-finite and $X \subseteq [\![Y, Z]\!]$. Then there exist A, B such that A is locally-finite and $A \subseteq Y$ and B is locally-finite and $B \subseteq Z$ and $X \subseteq [\![A, B]\!]$.

(27) Suppose X is locally-finite and Z is locally-finite and $X \subseteq [\![Y, Z]\!]$. Then there exists A such that A is locally-finite and $A \subseteq Y$ and $X \subseteq [\![A, Z]\!]$.

(28) Let M be a non-empty locally-finite many sorted set indexed by I. Suppose that for all many sorted sets A, B indexed by I such that $A \in M$ and $B \in M$ holds $A \subseteq B$ or $B \subseteq A$. Then there exists a many sorted set m indexed by I such that $m \in M$ and for every many sorted set K indexed by I such that $K \in M$ holds $m \subseteq K$.

(29) Let M be a non-empty locally-finite many sorted set indexed by I. Suppose that for all many sorted sets A, B indexed by I such that $A \in M$ and $B \in M$ holds $A \subseteq B$ or $B \subseteq A$. Then there exists a many sorted set m indexed by I such that $m \in M$ and for every many sorted set K indexed by I such that $K \in M$ holds $K \subseteq m$.

(30) If Z is locally-finite and $Z \subseteq \operatorname{rng}_\kappa F(\kappa)$, then there exists Y such that $Y \subseteq \operatorname{dom}_\kappa F(\kappa)$ and Y is locally-finite and $F \circ Y = Z$.

3. A Family of Subsets of Many Sorted Sets

Let us consider I, M.

(Def.1) A many sorted subset of 2^M is said to be a subset family of M.

Let us consider I, M. Note that there exists a subset family of M which is non-empty.

Let us consider I, M. Then 2^M is a subset family of M.

Let us consider I, M. One can check that there exists a subset family of M which is empty yielding and locally-finite.

One can prove the following proposition

(31) $\emptyset_I$ is an empty yielding locally-finite subset family of M.

Let us consider I and let M be a locally-finite many sorted set indexed by I. Note that there exists a subset family of M which is non-empty and locally-finite.

We follow the rules: S_1, S_2, S_3 will be subset families of M, S_4 will be a non-empty subset family of M, and V, W will be many sorted subsets of M.

Let I be a non empty set, let M be a many sorted set indexed by I, let S_1 be a subset family of M, and let i be an element of I. Then $S_1(i)$ is a family of subsets of $M(i)$.

The following propositions are true:

(32) If $i \in I$, then $S_1(i)$ is a family of subsets of $M(i)$.

(33) If $A \in S_1$, then A is a many sorted subset of M.

(34) $S_1 \cup S_2$ is a subset family of M.

(35) $S_1 \cap S_2$ is a subset family of M.

(36) $S_1 \setminus A$ is a subset family of M.

(37) $S_1 \dot{-} S_2$ is a subset family of M.

(38) If $A \subseteq M$, then $\{A\}$ is a subset family of M.

(39) If $A \subseteq M$ and $B \subseteq M$, then $\{A, B\}$ is a subset family of M.

(40) $\bigcup S_1 \subseteq M$.

4. Intersection of a Family of Many Sorted Sets

Let us consider I, M, S_1. The functor $\bigcap S_1$ yields a many sorted set indexed by I and is defined by:

(Def.2) For arbitrary i such that $i \in I$ there exists a family Q of subsets of $M(i)$ such that $Q = S_1(i)$ and $(\bigcap S_1)(i) = \operatorname{Intersect}(Q)$.

Let us consider I, M, S_1. Then $\bigcap S_1$ is a many sorted subset of M.

We now state a number of propositions:

(41) If $S_1 = \emptyset_I$, then $\bigcap S_1 = M$.

(42) $\bigcap S_4 \subseteq \bigcup S_4$.

(43) If $A \in S_1$, then $\bigcap S_1 \subseteq A$.

(44) If $\emptyset_I \in S_1$, then $\bigcap S_1 = \emptyset_I$.

(45) Let Z, M be many sorted sets indexed by I and let S_1 be a non-empty subset family of M. Suppose that for every many sorted set Z_1 indexed by I such that $Z_1 \in S_1$ holds $Z \subseteq Z_1$. Then $Z \subseteq \bigcap S_1$.

(46) If $S_1 \subseteq S_2$, then $\bigcap S_2 \subseteq \bigcap S_1$.

(47) If $A \in S_1$ and $A \subseteq B$, then $\bigcap S_1 \subseteq B$.

(48) If $A \in S_1$ and $A \cap B = \emptyset_I$, then $\bigcap S_1 \cap B = \emptyset_I$.

(49) If $S_3 = S_1 \cup S_2$, then $\bigcap S_3 = \bigcap S_1 \cap \bigcap S_2$.

(50) If $S_1 = \{V\}$, then $\bigcap S_1 = V$.

(51) If $S_1 = \{V, W\}$, then $\bigcap S_1 = V \cap W$.

(52) If $A \in \bigcap S_1$, then for every B such that $B \in S_1$ holds $A \in B$.

(53) Let A, M be many sorted sets indexed by I and let S_1 be a non-empty subset family of M. Suppose $A \in M$ and for every many sorted set B indexed by I such that $B \in S_1$ holds $A \in B$. Then $A \in \bigcap S_1$.

Let us consider I, M. A subset family of M is additive if:

(Def.3) For all A, B such that $A \in$ it and $B \in$ it holds $A \cup B \in$ it.

A subset family of M is absolutely-additive if:

(Def.4) For every subset family F of M such that $F \subseteq$ it holds $\bigcup F \in$ it.

A subset family of M is multiplicative if:

(Def.5) For all A, B such that $A \in$ it and $B \in$ it holds $A \cap B \in$ it.

A subset family of M is absolutely-multiplicative if:

(Def.6) For every subset family F of M such that $F \subseteq$ it holds $\bigcap F \in$ it.

A subset family of M is properly-upper-bound if:

(Def.7) $M \in$ it.

A subset family of M is properly-lower-bound if:

(Def.8) $\emptyset_I \in$ it.

Let us consider I, M. Observe that there exists a subset family of M which is non-empty additive absolutely-additive multiplicative absolutely-multiplicative properly-upper-bound and properly-lower-bound.

Let us consider I, M. Then 2^M is an additive absolutely-additive multiplicative absolutely-multiplicative properly-upper-bound properly-lower-bound subset family of M.

Let us consider I, M. Note that every subset family of M which is absolutely-additive is also additive.

Let us consider I, M. Note that every subset family of M which is absolutely-multiplicative is also multiplicative.

Let us consider I, M. One can check that every subset family of M which is absolutely-multiplicative is also properly-upper-bound.

Let us consider I, M. Observe that every subset family of M which is properly-upper-bound is also non-empty.

Let us consider I, M. Note that every subset family of M which is absolutely-additive is also properly-lower-bound.

Let us consider I, M. Note that every subset family of M which is properly-lower-bound is also non-empty.

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