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Natural Numbers

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The papers [6], [4], [2], [7], [1], [3], [5], and [8] provide the terminology and notation for this paper.

1. Preliminaries

In this article we present several logical schemes. The scheme *NonUniqRecExD* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every element n of $\mathbb{N}$ holds $\mathcal{P}[n, f(n), f(n+1)]$

provided the following condition is satisfied:

- For every natural number n and for every element x of $\mathcal{A}$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[n, x, y]$.

The scheme *NonUniqFinRecExD* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a natural number $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exists a finite sequence p of elements of $\mathcal{A}$ such that $\operatorname{len} p = \mathcal{C}$ but $p(1) = \mathcal{B}$ or $\mathcal{C} = 0$ but for every natural number n such that $1 \leqslant n$ and $n \leqslant \mathcal{C} - 1$ holds $\mathcal{P}[n, p(n), p(n+1)]$

provided the parameters meet the following requirement:

- Let n be a natural number. Suppose $1 \leqslant n$ and $n \leqslant \mathcal{C} - 1$. Let x be an element of $\mathcal{A}$. Then there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[n, x, y]$.

The scheme *NonUniqPiFinRecExD* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a natural number $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

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There exists a finite sequence p of elements of $\mathcal{A}$ such that $\operatorname{len} p = \mathcal{C}$ but $\pi_1 p = \mathcal{B}$ or $\mathcal{C} = 0$ but for every natural number n such that $1 \leqslant n$ and $n \leqslant \mathcal{C} - 1$ holds $\mathcal{P}[n, \pi_n p, \pi_{n+1} p]$

provided the following condition is met:

- Let n be a natural number. Suppose $1 \leqslant n$ and $n \leqslant \mathcal{C} - 1$. Let x be an element of $\mathcal{A}$. Then there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[n, x, y]$.

The following two propositions are true:

(1) For every real number x holds $x < \lfloor x \rfloor + 1$.

(2) For all real numbers x, y such that $x \geqslant 0$ and $y > 0$ holds $\frac{x}{\lfloor \frac{x}{y} \rfloor + 1} < y$.

2. Division and Rest of Division

The following propositions are true:

(3) For every natural number n holds n is empty iff $n = 0$.

(4) For every natural number n holds $0 \div n = 0$.

(5) For every non empty natural number n holds $n \div n = 1$.

(6) For every natural number n holds $n \div 1 = n$.

(7) For all natural numbers i, j, k, l such that $i \leqslant j$ and $k \leqslant j$ holds if $i = (j -' k) + l$, then $k = (j -' i) + l$.

(8) For all natural numbers i, n such that $i \in \operatorname{Seg} n$ holds $(n -' i) + 1 \in \operatorname{Seg} n$.

(9) For all natural numbers i, j such that $j < i$ holds $(i -' (j+1)) + 1 = i -' j$.

(10) For all natural numbers i, j such that $i \geqslant j$ holds $j -' i = 0$.

(11) For all non empty natural numbers i, j holds $i -' j < i$.

(12) Let n, k be natural numbers. Suppose $k \leqslant n$. Then the n-th power of 2 = (the k-th power of 2) $\cdot$(the $(n -' k)$-th power of 2).

(13) For all natural numbers n, k such that $k \leqslant n$ holds the k-th power of 2 $\mid$ the n-th power of 2.

(14) For all natural numbers n, k such that $k > 0$ and $n \div k = 0$ holds $n < k$.

(15) For all natural numbers n, k such that $k > 0$ and $k \leqslant n$ holds $n \div k \geqslant 1$.

(16) For all natural numbers n, k such that $k \neq 0$ holds $(n+k) \div k = (n \div k) + 1$.

(17) For all natural numbers n, k, i such that $k \mid n$ and $1 \leqslant n$ and $1 \leqslant i$ and $i \leqslant k$ holds $(n -' i) \div k = (n \div k) - 1$.

(18) Let n, k be natural numbers. Suppose $k \leqslant n$. Then (the n-th power of 2) $\div$(the k-th power of 2) = the $(n -' k)$-th power of 2.

(19) For every natural number n such that $n > 0$ holds (the n-th power of 2) $\operatorname{mod} 2 = 0$.

(20) For every natural number n such that $n > 0$ holds $n \bmod 2 = 0$ iff $(n -' 1) \bmod 2 = 1$.

(21) For every non empty natural number n such that $n \neq 1$ holds $n > 1$.

(22) For all natural numbers n, k such that $n \leqslant k$ and $k < n + n$ holds $k \div n = 1$.

(23) For every natural number n holds n is even iff $n \bmod 2 = 0$.

(24) For every natural number n holds n is odd iff $n \bmod 2 = 1$.

(25) For all natural numbers n, k, t such that $1 \leqslant t$ and $k \leqslant n$ and $2 \cdot t \mid k$ holds $n \div t$ is even iff $(n -' k) \div t$ is even.

(26) For all natural numbers n, m, k such that $n \leqslant m$ holds $n \div k \leqslant m \div k$.

(27) For all natural numbers n, k such that $k \leqslant 2 \cdot n$ holds $(k + 1) \div 2 \leqslant n$.

(28) For every even natural number n holds $n \div 2 = (n + 1) \div 2$.

(29) For all natural numbers n, k, i holds $n \div k \div i = n \div k \cdot i$.

Let n be a natural number. We say that n is non trivial if and only if:

(Def. 1) $n \neq 0$ and $n \neq 1$.

One can verify that every natural number which is non trivial is also non empty.

One can check that there exists a natural number which is non trivial.

The following two propositions are true:

(30) For every natural number k holds k is non trivial iff k is non empty and $k \neq 1$.

(31) For every non trivial natural number k holds $k \geqslant 2$.

The scheme *Ind from 2* concerns a unary predicate $\mathcal{P}$, and states that:

For every non trivial natural number k holds $\mathcal{P}[k]$

provided the following conditions are met:

- $\mathcal{P}[2]$, and
- For every non trivial natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$.

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