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Oriented Chains

Yatsuka Nakamura
Shinshu University
Nagano

Piotr Rudnicki
University of Alberta
Edmonton

Summary. In [5] we introduced a number of notions about vertex sequences associated with undirected chains of edges in graphs. In this article, we introduce analogous concepts for oriented chains and use them to prove properties of cutting and glueing of oriented chains, and the existence of a simple oriented chain in an oriented chain.

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The notation and terminology used here are introduced in the following papers: [6], [8], [2], [3], [4], [5], [1], [9], and [7].

1. Oriented Vertex Sequences

For simplicity, we adopt the following rules: p, q denote finite sequences, m, n denote natural numbers, G denotes a graph, x, y, v, v_1, v_2, v_3, v_4 denote elements of the vertices of G, e denotes a set, and X denotes a set.

Let us consider G, let us consider x, y, and let us consider e. We say that e orientedly joins x, y if and only if:

(Def. 1) (The source of $G)(e) = x$ and (the target of $G)(e) = y$.

We now state the proposition

(1) If e orientedly joins v_1, v_2, then e joins v_1 with v_2.

Let us consider G and let x, y be elements of the vertices of G. We say that x, y are orientedly incident if and only if:

(Def. 2) There exists a set v such that $v \in$ the edges of G and v orientedly joins x, y.

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One can prove the following proposition

(2) If e orientedly joins v_1, v_2 and e orientedly joins v_3, v_4, then $v_1 = v_3$ and $v_2 = v_4$.

We follow the rules: v_5, v_6, v_7 are finite sequences of elements of the vertices of G and c, c_1, c_2 are oriented chains of G.

We now state the proposition

(3) ε is an oriented chain of G.

Let us consider G. Observe that there exists a chain of G which is empty and oriented.

Let us consider G, X. The functor G-SVSet X yields a set and is defined by:

(Def. 3) $G\text{-SVSet}\, X = \{v : \bigvee_{e:\,\text{element of the edges of } G} (e \in X \ \wedge\ v = (\text{the source of } G)(e))\}$.

Let us consider G, X. The functor G-TVSet X yielding a set is defined by:

(Def. 4) $G\text{-TVSet}\, X = \{v : \bigvee_{e:\,\text{element of the edges of } G} (e \in X \ \wedge\ v = (\text{the target of } G)(e))\}$.

Next we state the proposition

(4) If $X = \emptyset$, then $G\text{-SVSet}\, X = \emptyset$ and $G\text{-TVSet}\, X = \emptyset$.

Let us consider G, v_5 and let c be a finite sequence. We say that v_5 is oriented vertex seq of c if and only if:

(Def. 5) $\operatorname{len} v_5 = \operatorname{len} c + 1$ and for every n such that $1 \leqslant n$ and $n \leqslant \operatorname{len} c$ holds $c(n)$ orientedly joins $\pi_n v_5$, $\pi_{n+1} v_5$.

One can prove the following propositions:

(5) If v_5 is oriented vertex seq of c, then v_5 is vertex sequence of c.

(6) If v_5 is oriented vertex seq of c, then $G\text{-SVSet}\operatorname{rng} c \subseteq \operatorname{rng} v_5$.

(7) If v_5 is oriented vertex seq of c, then $G\text{-TVSet}\operatorname{rng} c \subseteq \operatorname{rng} v_5$.

(8) If $c \neq \varepsilon$ and v_5 is oriented vertex seq of c, then $\operatorname{rng} v_5 \subseteq (G\text{-SVSet}\operatorname{rng} c) \cup (G\text{-TVSet}\operatorname{rng} c)$.

2. Cutting and Glueing of Oriented Chains

One can prove the following propositions:

(9) $\langle v \rangle$ is oriented vertex seq of ε.

(10) There exists v_5 such that v_5 is oriented vertex seq of c.

(11) If $c \neq \varepsilon$ and v_6 is oriented vertex seq of c and v_7 is oriented vertex seq of c, then $v_6 = v_7$.

Let us consider G, c. Let us assume that $c \neq \varepsilon$. The functor oriented-vertex-seq c yielding a finite sequence of elements of the vertices of G is defined as follows:

(Def. 6) oriented-vertex-seq c is oriented vertex seq of c.

Next we state several propositions:

(12) If v_5 is oriented vertex seq of c and $c_1 = c \restriction \mathrm{Seg}\, n$ and $v_6 = v_5 \restriction \mathrm{Seg}(n+1)$, then v_6 is oriented vertex seq of c_1.

(13) If $1 \leqslant m$ and $m \leqslant n$ and $n \leqslant \mathrm{len}\, c$ and $q = \langle c(m), \ldots, c(n) \rangle$, then q is an oriented chain of G.

(14) Suppose $1 \leqslant m$ and $m \leqslant n$ and $n \leqslant \mathrm{len}\, c$ and $c_1 = \langle c(m), \ldots, c(n) \rangle$ and v_5 is oriented vertex seq of c and $v_6 = \langle v_5(m), \ldots, v_5(n+1) \rangle$. Then v_6 is oriented vertex seq of c_1.

(15) Suppose v_6 is oriented vertex seq of c_1 and v_7 is oriented vertex seq of c_2 and $v_6(\mathrm{len}\, v_6) = v_7(1)$. Then $c_1 \frown c_2$ is an oriented chain of G.

(16) Suppose v_6 is oriented vertex seq of c_1 and v_7 is oriented vertex seq of c_2 and $v_6(\mathrm{len}\, v_6) = v_7(1)$ and $c = c_1 \frown c_2$ and $v_5 = v_6 \frown\frown v_7$. Then v_5 is oriented vertex seq of c.

3. Oriented Simple Chains in Oriented Chains

Let us consider G and let I_1 be an oriented chain of G. We say that I_1 is Simple if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists v_5 such that v_5 is oriented vertex seq of I_1 and for all n, m such that $1 \leqslant n$ and $n < m$ and $m \leqslant \mathrm{len}\, v_5$ and $v_5(n) = v_5(m)$ holds $n = 1$ and $m = \mathrm{len}\, v_5$.

Let us consider G. Note that there exists an oriented chain of G which is Simple.

Let us consider G. One can verify that there exists a chain of G which is oriented and simple.

Next we state two propositions:

(17) Every oriented simple chain of G is an oriented chain of G.

(18) For every oriented chain q of G holds $q \restriction \mathrm{Seg}\, n$ is an oriented chain of G.

In the sequel s_1 is an oriented simple chain of G.

Next we state several propositions:

(19) $s_1 \restriction \mathrm{Seg}\, n$ is an oriented simple chain of G.

(20) For every oriented chain s_1' of G such that $s_1' = s_1$ holds s_1' is Simple.

(21) Every Simple oriented chain of G is an oriented simple chain of G.

(22) Suppose c is not Simple and v_5 is oriented vertex seq of c. Then there exists a FinSubsequence f_1 of c and there exists a FinSubsequence f_2 of v_5 and there exist c_1, v_6 such that $\mathrm{len}\, c_1 < \mathrm{len}\, c$ and v_6 is oriented vertex seq of c_1 and $\mathrm{len}\, v_6 < \mathrm{len}\, v_5$ and $v_5(1) = v_6(1)$ and $v_5(\mathrm{len}\, v_5) = v_6(\mathrm{len}\, v_6)$ and $\mathrm{Seq}\, f_1 = c_1$ and $\mathrm{Seq}\, f_2 = v_6$.

(23) Suppose v_5 is oriented vertex seq of c. Then there exists a FinSubsequence f_1 of c and there exists a FinSubsequence f_2 of v_5 and there exist s_1, v_6 such that $\mathrm{Seq}\, f_1 = s_1$ and $\mathrm{Seq}\, f_2 = v_6$ and v_6 is oriented vertex seq of s_1 and $v_5(1) = v_6(1)$ and $v_5(\mathrm{len}\, v_5) = v_6(\mathrm{len}\, v_6)$.

Let us consider G. Observe that every oriented chain of G which is empty is also oriented.

Next we state three propositions:

(24) If p is an oriented path of G, then $p \restriction \mathrm{Seg}\, n$ is an oriented path of G.

(25) s_1 is an oriented path of G.

(26) Let c_1 be a finite sequence. Then
(i) c_1 is a Simple oriented chain of G iff c_1 is an oriented simple chain of G, and
(ii) if c_1 is an oriented simple chain of G, then c_1 is an oriented path of G.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Krzysztof Hryniewiecki. Graphs. *Formalized Mathematics*, 2(**3**):365–370, 1991.
[5] Yatsuka Nakamura and Piotr Rudnicki. Vertex sequences induced by chains. *Formalized Mathematics*, 5(**3**):297–304, 1996.
[6] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[7] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[8] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[9] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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