The Differentiable Functions on Normed Linear Spaces

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Summary. In this article, the basic properties of the differentiable functions on normed linear spaces are described.

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The notation and terminology used in this paper are introduced in the following papers: [20], [23], [4], [24], [6], [5], [19], [3], [10], [1], [18], [7], [21], [22], [11], [8], [9], [25], [13], [15], [16], [17], [12], [14], and [2].

For simplicity, we adopt the following rules: $n, k$ denote natural numbers, $x, X, Z$ denote sets, $g, r$ denote real numbers, $S$ denotes a real normed space, $r_1$ denotes a sequence of real numbers, $s_1, s_2$ denote sequences of $S$, $x_0$ denotes a point of $S$, and $Y$ denotes a subset of $S$.

Next we state several propositions:

1. For every point $x_0$ of $S$ and for all neighbourhoods $N_1, N_2$ of $x_0$ there exists a neighbourhood $N$ of $x_0$ such that $N \subseteq N_1$ and $N \subseteq N_2$.

2. Let $X$ be a subset of $S$. Suppose $X$ is open. Let $r$ be a point of $S$. If $r \in X$, then there exists a neighbourhood $N$ of $r$ such that $N \subseteq X$.

3. Let $X$ be a subset of $S$. Suppose $X$ is open. Let $r$ be a point of $S$. If $r \in X$, then there exists $g$ such that $0 < g$ and $\{y; y \text{ ranges over points of } S: \|y - r\| < g\} \subseteq X$.

4. Let $X$ be a subset of $S$. Suppose that for every point $r$ of $S$ such that $r \in X$ there exists a neighbourhood $N$ of $r$ such that $N \subseteq X$. Then $X$ is open.

5. Let $X$ be a subset of $S$. Then for every point $r$ of $S$ such that $r \in X$ there exists a neighbourhood $N$ of $r$ such that $N \subseteq X$ if and only if $X$ is open.
Let $S$ be a zero structure and let $f$ be a sequence of $S$. We say that $f$ is non-zero if and only if:

(Def. 1) $\text{rng} \ f \subseteq (\text{the carrier of } S) \setminus \{0_S\}$.

We introduce $f$ is non-zero as a synonym of $f$ is non-zero.

We now state two propositions:

(6) $s_1$ is non-zero iff for every $x$ such that $x \in \mathbb{N}$ holds $s_1(x) \neq 0_S$.
(7) $s_1$ is non-zero iff for every $n$ holds $s_1(n) \neq 0_S$.

Let $R_1$ be a real linear space, let $S$ be a sequence of $R_1$, and let $a$ be a sequence of real numbers. The functor $a \cdot S$ yields a sequence of $R_1$ and is defined as follows:

(Def. 2) For every $n$ holds $(a \cdot S)(n) = a(n) \cdot S(n)$.

Let $R_1$ be a real linear space, let $z$ be a point of $R_1$, and let $a$ be a sequence of real numbers. The functor $a \cdot z$ yields a sequence of $R_1$ and is defined by:

(Def. 3) For every $n$ holds $(a \cdot z)(n) = a(n) \cdot z$.

Next we state a number of propositions:

(8) For all sequences $r_2, r_3$ of real numbers holds $(r_2 + r_3) \cdot s_1 = r_2 \cdot s_1 + r_3 \cdot s_1$.
(9) For every sequence $r_1$ of real numbers and for all sequences $s_2, s_3$ of $S$ holds $r_1 \cdot (s_2 + s_3) = r_1 \cdot s_2 + r_1 \cdot s_3$.
(10) For every sequence $r_1$ of real numbers holds $r \cdot (r_1 \cdot s_1) = r_1 \cdot (r \cdot s_1)$.
(11) For all sequences $r_2, r_3$ of real numbers holds $(r_2 - r_3) \cdot s_1 = r_2 \cdot s_1 - r_3 \cdot s_1$.
(12) For every sequence $r_1$ of real numbers and for all sequences $s_2, s_3$ of $S$ holds $r_1 \cdot (s_2 - s_3) = r_1 \cdot s_2 - r_1 \cdot s_3$.
(13) If $r_1$ is convergent and $s_1$ is convergent, then $r_1 \cdot s_1$ is convergent.
(14) If $r_1$ is convergent and $s_1$ is convergent, then $\lim(r_1 \cdot s_1) = \lim r_1 \cdot \lim s_1$.
(15) $(s_1 + s_2) \uparrow k = s_1 \uparrow k + s_2 \uparrow k$.
(16) $(s_1 - s_2) \uparrow k = s_1 \uparrow k - s_2 \uparrow k$.
(17) If $s_1$ is non-zero, then $s_1 \uparrow k$ is non-zero.
(18) $s_1 \uparrow k$ is a subsequence of $s_1$.
(19) If $s_1$ is constant and $s_2$ is a subsequence of $s_1$, then $s_2$ is constant.
(20) If $s_1$ is constant and $s_2$ is a subsequence of $s_1$, then $s_1 = s_2$.

Let us consider $S$ and let $I_1$ be a sequence of $S$. We say that $I_1$ is convergent to $0$ if and only if:

(Def. 4) $I_1$ is non-zero and convergent and $\lim I_1 = 0_S$.

The following propositions are true:

(21) Let $X$ be a real normed space and $s_1$ be a sequence of $X$. Suppose $s_1$ is constant. Then $s_1$ is convergent and for every natural number $k$ holds $\lim s_1 = s_1(k)$. 

For every real number \( r \) such that \( 0 < r \) and for every \( n \) holds \( s_1(n) = \frac{1}{n+r} \cdot x_0 \) holds \( s_1 \) is convergent.

For every real number \( r \) such that \( 0 < r \) and for every \( n \) holds \( s_1(n) = \frac{1}{n+r} \cdot x_0 \) holds \( \lim s_1 = 0_S \).

Let \( a \) be a convergent to 0 sequence of real numbers and \( z \) be a point of \( S \). If \( z \neq 0_S \), then \( a \cdot z \) is convergent to 0.

For every point \( r \) of \( S \) holds \( r \in Y \) iff \( r \in \text{carrier of } S \) iff \( Y = \text{carrier of } S \).

For simplicity, we adopt the following rules: \( S, T \) denote non trivial real normed spaces, \( f, f_1, f_2 \) denote partial functions from \( S \) to \( T \), \( s_4, s_1 \) denote sequences of \( S \), and \( x_0 \) denotes a point of \( S \).

Let \( S \) be a non trivial real normed space. Note that there exists a sequence of \( S \) which is convergent to 0.

Let us consider \( S \). Note that there exists a sequence of \( S \) which is constant.

In the sequel \( h \) is a convergent to 0 sequence of \( S \) and \( c \) is a constant sequence of \( S \).

Let us consider \( S, T \) and let \( I_1 \) be a partial function from \( S \) to \( T \). We say that \( I_1 \) is rest-like if and only if:

(Def. 5) \( I_1 \) is total and for every \( h \) holds \( \|h\|^{-1} (I_1 \cdot h) \) is convergent and \( \lim(\|h\|^{-1} (I_1 \cdot h)) = 0_T \).

Let us consider \( S, T \). Observe that there exists a partial function from \( S \) to \( T \) which is rest-like.

Let us consider \( S, T \). A rest of \( S, T \) is a rest-like partial function from \( S \) to \( T \).

We now state two propositions:

(26) Let \( R \) be a partial function from \( S \) to \( T \). Suppose \( R \) is total. Then \( R \) is rest-like if and only if for every real number \( r \) such that \( r > 0 \) there exists a real number \( d \) such that \( d > 0 \) and for every point \( z \) of \( S \) such that \( z \neq 0_S \) and \( \|z\| < d \) holds \( \|z\|^{-1} \cdot \|Rz\| < r \).

(27) For every rest \( R \) of \( S, T \) and for every convergent to 0 sequence \( s \) of \( S \) holds \( R \cdot s \) is convergent and \( \lim(R \cdot s) = 0_T \).

In the sequel \( R, R_2, R_3 \) are rests of \( S, T \) and \( L \) is a point of \( R\text{NormSpaceOfBoundedLinearOperators}(S,T) \).

Next we state several propositions:

(28) \( \text{rng}(s_1 \uparrow n) \subseteq \text{rng } s_1 \).

(29) For every partial function \( h \) from \( S \) to \( T \) and for every sequence \( s_1 \) of \( S \) such that \( \text{rng } s_1 \subseteq \text{dom } h \) holds \( (h \cdot s_1) \uparrow n = h \cdot (s_1 \uparrow n) \).

(30) Let \( h_1, h_2 \) be partial functions from \( S \) to \( T \) and \( s_1 \) be a sequence of \( S \). If \( h_1 \) is total and \( h_2 \) is total, then \( (h_1 + h_2) \cdot s_1 = h_1 \cdot s_1 + h_2 \cdot s_1 \) and \( (h_1 - h_2) \cdot s_1 = h_1 \cdot s_1 - h_2 \cdot s_1 \).
(31) Let $h$ be a partial function from $S$ to $T$, $s_1$ be a sequence of $S$, and $r$ be a real number. If $h$ is total, then $(r \cdot h) \cdot s_1 = r \cdot (h \cdot s_1)$.

(32) $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:

(i) $x_0 \in \text{dom } f$, and

(ii) for every sequence $s_4$ of $S$ such that $\text{rng } s_4 \subseteq \text{dom } f$ and $s_4$ is convergent and $\lim s_4 = x_0$ and for every $n$ holds $s_4(n) \neq x_0$ holds $f \cdot s_4$ is convergent and $f_{x_0} = \lim(f \cdot s_4)$.

(33) For all $R_2$, $R_3$ holds $R_2 + R_3$ is a rest of $S$, $T$ and $R_2 - R_3$ is a rest of $S$, $T$.

(34) For all $r$, $R$ holds $r R$ is a rest of $S$, $T$.

Let us consider $S$, $T$, let $f$ be a partial function from $S$ to $T$, and let $x_0$ be a point of $S$. We say that $f$ is differentiable in $x_0$ if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a neighbourhood $N$ of $x_0$ such that $N \subseteq \text{dom } f$ and there exist $L, R$ such that for every point $x$ of $S$ such that $x \in N$ holds $f_x - f_{x_0} = L(x - x_0) + R_{x-x_0}$.

Let us consider $S$, $T$, let $f$ be a partial function from $S$ to $T$, and let $x_0$ be a point of $S$. Let us assume that $f$ is differentiable in $x_0$. The functor $f'(x_0)$ yielding a point of $\text{RNNormSpaceOfBoundedLinearOperators}(S, T)$ is defined by the condition (Def. 7).

(Def. 7) There exists a neighbourhood $N$ of $x_0$ such that $N \subseteq \text{dom } f$ and there exists $R$ such that for every point $x$ of $S$ such that $x \in N$ holds $f_x - f_{x_0} = f'(x_0)(x - x_0) + R_{x-x_0}$.

Let us consider $X$, let us consider $S$, $T$, and let $f$ be a partial function from $S$ to $T$. We say that $f$ is differentiable on $X$ if and only if:

(Def. 8) $X \subseteq \text{dom } f$ and for every point $x$ of $S$ such that $x \in X$ holds $f | X$ is differentiable in $x$.

Next we state three propositions:

(35) Let $f$ be a partial function from $S$ to $T$. If $f$ is differentiable on $X$, then $X$ is a subset of the carrier of $S$.

(36) Let $f$ be a partial function from $S$ to $T$ and $Z$ be a subset of $S$. Suppose $Z$ is open. Then $f$ is differentiable on $Z$ if and only if the following conditions are satisfied:

(i) $Z \subseteq \text{dom } f$, and

(ii) for every point $x$ of $S$ such that $x \in Z$ holds $f$ is differentiable in $x$.

(37) Let $f$ be a partial function from $S$ to $T$ and $Y$ be a subset of $S$. If $f$ is differentiable on $Y$, then $Y$ is open.

Let us consider $S$, $T$, let $f$ be a partial function from $S$ to $T$, and let $X$ be a set. Let us assume that $f$ is differentiable on $X$. The functor $f'_{X}$ yielding
a partial function from $S$ to $\text{RNormSpaceOfBoundedLinearOperators}(S,T)$ is defined by:

(Def. 9) $\text{dom}(f'_x) = X$ and for every point $x$ of $S$ such that $x \in X$ holds $(f'_x)_x = f'(x)$.

One can prove the following proposition

(38) Let $f$ be a partial function from $S$ to $T$ and $Z$ be a subset of $S$. Suppose $Z$ is open and $Z \subseteq \text{dom } f$ and there exists a point $r$ of $T$ such that $\text{rng } f = \{r\}$. Then $f$ is differentiable on $Z$ and for every point $x$ of $S$ such that $x \in Z$ holds $(f'_x)_x = 0_{\text{RNormSpaceOfBoundedLinearOperators}(S,T)}$.

Let us consider $S$ and let us consider $h, n$. Observe that $h \uparrow n$ is convergent to 0.

Let us consider $S$ and let us consider $c, n$. Observe that $c \uparrow n$ is constant.

The following propositions are true:

(39) Let $x_0$ be a point of $S$ and $N$ be a neighbourhood of $x_0$. Suppose $f$ is differentiable in $x_0$ and $N \subseteq \text{dom } f$. Let $h$ be a convergent to 0 sequence of $S$ and given $c$. If $\text{rng } c = \{x_0\}$ and $\text{rng}(h + c) \subseteq N$, then $f \cdot (h + c) - f \cdot c$ is convergent and $\text{lim}(f \cdot (h + c) - f \cdot c) = 0_T$.

(40) Let given $f_1, f_2, x_0$. Suppose $f_1$ is differentiable in $x_0$ and $f_2$ is differentiable in $x_0$. Then $f_1 + f_2$ is differentiable in $x_0$ and $(f_1 + f_2)'(x_0) = f_1'(x_0) + f_2'(x_0)$.

(41) Let given $f_1, f_2, x_0$. Suppose $f_1$ is differentiable in $x_0$ and $f_2$ is differentiable in $x_0$. Then $f_1 - f_2$ is differentiable in $x_0$ and $(f_1 - f_2)'(x_0) = f_1'(x_0) - f_2'(x_0)$.

(42) For all $r, f, x_0$ such that $f$ is differentiable in $x_0$ holds $rf$ is differentiable in $x_0$ and $(rf)'(x_0) = r \cdot f'(x_0)$.

(43) Let $f$ be a partial function from $S$ to $S$ and $Z$ be a subset of $S$. Suppose $Z$ is open and $Z \subseteq \text{dom } f$ and $f\mid Z = \text{id}_Z$. Then $f$ is differentiable on $Z$ and for every point $x$ of $S$ such that $x \in Z$ holds $(f\mid Z)_x = \text{id}_{\text{the carrier of } S}$.

(44) Let $Z$ be a subset of $S$. Suppose $Z$ is open. Let given $f_1, f_2$. Suppose $Z \subseteq \text{dom}(f_1 + f_2)$ and $f_1$ is differentiable on $Z$ and $f_2$ is differentiable on $Z$. Then $f_1 + f_2$ is differentiable on $Z$ and for every point $x$ of $S$ such that $x \in Z$ holds $((f_1 + f_2)'_Z)_x = f_1'(x) + f_2'(x)$.

(45) Let $Z$ be a subset of $S$. Suppose $Z$ is open. Let given $f_1, f_2$. Suppose $Z \subseteq \text{dom}(f_1 - f_2)$ and $f_1$ is differentiable on $Z$ and $f_2$ is differentiable on $Z$. Then $f_1 - f_2$ is differentiable on $Z$ and for every point $x$ of $S$ such that $x \in Z$ holds $((f_1 - f_2)'_Z)_x = f_1'(x) - f_2'(x)$.

(46) Let $Z$ be a subset of $S$. Suppose $Z$ is open. Let given $r, f$. Suppose $Z \subseteq \text{dom}(rf)$ and $f$ is differentiable on $Z$. Then $rf$ is differentiable on $Z$ and for every point $x$ of $S$ such that $x \in Z$ holds $((rf)'_Z)_x = r \cdot f'(x)$.

(47) Let $Z$ be a subset of $S$. Suppose $Z$ is open. Suppose $Z \subseteq \text{dom } f$ and $f$
is a constant on $Z$. Then $f$ is differentiable on $Z$ and for every point $x$ of $S$ such that $x \in Z$ holds $(f|_Z)_x = 0_{\text{RNormSpaceOfBoundedLinearOperators}(S,T)}$.

Let $f$ be a partial function from $S$ to $S$, $r$ be a real number, $p$ be a point of $S$, and $Z$ be a subset of $S$. Suppose $Z$ is open. Suppose $Z \subseteq \text{dom } f$ and for every point $x$ of $S$ such that $x \in Z$ holds $f_x = r \cdot x + p$. Then $f$ is differentiable on $Z$ and for every point $x$ of $S$ such that $x \in Z$ holds $(f|_Z)_x = r \cdot \text{FuncUnit}(S)$.

For every point $x_0$ of $S$ such that $f$ is differentiable in $x_0$ holds $f$ is continuous in $x_0$.

If $f$ is differentiable on $X$, then $f$ is continuous on $X$.

For every subset $Z$ of $S$ such that $Z$ is open holds if $f$ is differentiable on $X$ and $Z \subseteq X$, then $f$ is differentiable on $Z$.

Suppose $f$ is differentiable in $x_0$. Then there exists a neighbourhood $N$ of $x_0$ such that

(i) $N \subseteq \text{dom } f$, and

(ii) there exists $R$ such that $R_{0_S} = 0_T$ and $R$ is continuous in $0_S$ and for every point $x$ of $S$ such that $x \in N$ holds $f_x - f_{x_0} = f'(x_0)(x - x_0) + R_{x - x_0}$.

References


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