The Continuous Functions on Normed Linear Spaces

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Summary. In this article, the basic properties of the continuous function on normed linear spaces are described.

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The articles [16], [19], [20], [2], [21], [4], [9], [3], [1], [11], [15], [5], [17], [18], [10], [7], [8], [6], [13], [22], [12], and [14] provide the notation and terminology for this paper.

We use the following convention: \( n \) is a natural number, \( x, X, X_1 \) are sets, and \( s, r, p \) are real numbers.

Let \( S, T \) be 1-sorted structures. A partial function from \( S \) to \( T \) is a partial function from the carrier of \( S \) to the carrier of \( T \).

For simplicity, we adopt the following rules: \( S, T \) denote real normed spaces, \( f, f_1, f_2 \) denote partial functions from \( S \) to \( T \), \( s_1 \) denotes a sequence of \( S \), \( x_0, x_1, x_2 \) denote points of \( S \), and \( Y \) denotes a subset of \( S \).

Let \( R_1 \) be a real linear space and let \( S_1 \) be a sequence of \( R_1 \). The functor \(-S_1\) yields a sequence of \( R_1 \) and is defined as follows:

(Def. 1) For every \( n \) holds \((-S_1)(n) = -S_1(n)\).

Next we state two propositions:

(1) For all sequences \( s_2, s_3 \) of \( S \) holds \( s_2 - s_3 = s_2 + (-s_3) \).

(2) For every sequence \( s_4 \) of \( S \) holds \(-s_4 = (-1) \cdot s_4 \).

Let us consider \( S, T \) and let \( f \) be a partial function from \( S \) to \( T \). The functor \( \|f\| \) yielding a partial function from the carrier of \( S \) to \( \mathbb{R} \) is defined as follows:

(Def. 2) \( \text{dom} \|f\| = \text{dom} f \) and for every point \( c \) of \( S \) such that \( c \in \text{dom} \|f\| \) holds \( \|f\|(c) = \|f_c\| \).
Let us consider $S$, $x_0$. A subset of $S$ is called a neighbourhood of $x_0$ if:

(Def. 3) There exists a real number $g$ such that $0 < g$ and \{ $y$; $y$ ranges over points of $S$: $\|y - x_0\| < g$ $\}$ $\subseteq$ it.

The following two propositions are true:

(3) For every real number $g$ such that $0 < g$ holds \{ $y$; $y$ ranges over points of $S$: $\|y - x_0\| < g$ $\}$ is a neighbourhood of $x_0$.

(4) For every neighbourhood $N$ of $x_0$ holds $x_0 \in N$.

Let us consider $S$ and let $X$ be a subset of $S$. We say that $X$ is compact if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let $s_1$ be a sequence of $S$. Suppose $\text{rng} s_1 \subseteq X$. Then there exists a sequence $s_5$ of $S$ such that $s_5$ is a subsequence of $s_1$ and convergent and $\lim s_5 \in X$.

Let us consider $S$ and let $X$ be a subset of $S$. We say that $X$ is closed if and only if:

(Def. 5) For every sequence $s_1$ of $S$ such that $\text{rng} s_1 \subseteq X$ and $s_1$ is convergent holds $\lim s_1 \in X$.

Let us consider $S$ and let $X$ be a subset of $S$. We say that $X$ is open if and only if:

(Def. 6) $X^c$ is closed.

Let us consider $S$, $T$, let us consider $f$, and let $s_4$ be a sequence of $S$. Let us assume that $\text{rng} s_4 \subseteq \text{dom} f$. The functor $f \cdot s_4$ yields a sequence of $T$ and is defined as follows:

(Def. 7) $f \cdot s_4 = (f \text{ qua function}) \cdot (s_4)$.

Let us consider $S$, let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$, and let $s_4$ be a sequence of $S$. Let us assume that $\text{rng} s_4 \subseteq \text{dom} f$. The functor $f \cdot s_4$ yields a sequence of real numbers and is defined as follows:

(Def. 8) $f \cdot s_4 = (f \text{ qua function}) \cdot (s_4)$.

Let us consider $S$, $T$ and let us consider $f$, $x_0$. We say that $f$ is continuous in $x_0$ if and only if:

(Def. 9) $x_0 \in \text{dom} f$ and for every $s_1$ such that $\text{rng} s_1 \subseteq \text{dom} f$ and $s_1$ is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim (f \cdot s_1)$.

Let us consider $S$, let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$, and let us consider $x_0$. We say that $f$ is continuous in $x_0$ if and only if:

(Def. 10) $x_0 \in \text{dom} f$ and for every $s_1$ such that $\text{rng} s_1 \subseteq \text{dom} f$ and $s_1$ is convergent and $\lim s_1 = x_0$ holds $f \cdot s_1$ is convergent and $f_{x_0} = \lim (f \cdot s_1)$.

The scheme $\text{SeqPointNormSpChoice}$ deals with a non empty normed structure $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a sequence $s_1$ of $\mathcal{A}$ such that for every natural number $n$ holds $\mathcal{P}[n, s_1(n)]$
provided the following condition is met:

- For every natural number \( n \) there exists a point \( r \) of \( A \) such that \( P[n, r] \).

The following propositions are true:

1. For every sequence \( s_4 \) of \( S \) and for every partial function \( h \) from \( S \) to \( T \) such that \( \operatorname{rng} s_4 \subseteq \text{dom} \ h \), \( s_4(n) \in \text{dom} \ h. \)
2. For every sequence \( s_4 \) of \( S \) and for every set \( x \) holds \( x \in \operatorname{rng} s_4 \) iff there exists \( n \) such that \( x = s_4(n) \).
3. For all sequences \( s_4, s_2 \) of \( S \) such that \( s_2 \) is a subsequence of \( s_4 \) holds \( \operatorname{rng} s_2 \subseteq \operatorname{rng} s_4 \).
4. For all \( f, s_1 \) such that \( \operatorname{rng} s_1 \subseteq \text{dom} \ f \) and for every \( n \) holds \( (f \cdot s_1)(n) = f_{s_1(n)} \).
5. Let \( f \) be a partial function from the carrier of \( S \) to \( \mathbb{R} \) and given \( s_1 \). If \( \operatorname{rng} s_1 \subseteq \text{dom} \ f \), then for every \( n \) holds \( (f \cdot s_1)(n) = f_{s_1(n)} \).
6. Let \( h \) be a partial function from \( S \) to \( T \), \( s_4 \) be a sequence of \( S \), and \( N_1 \) be an increasing sequence of naturals. If \( \operatorname{rng} s_4 \subseteq \text{dom} \ h \), then \( (h \cdot s_4) \cdot N_1 = h \cdot (s_4 \cdot N_1) \).
7. Let \( h \) be a partial function from the carrier of \( S \) to \( \mathbb{R} \), \( s_4 \) be a sequence of \( S \), and \( N_1 \) be an increasing sequence of naturals. If \( \operatorname{rng} s_4 \subseteq \text{dom} \ h \), then \( (h \cdot s_4) \cdot N_1 = h \cdot (s_4 \cdot N_1) \).
8. Let \( h \) be a partial function from the carrier of \( S \) to \( \mathbb{R} \) and \( s_2, s_3 \) be sequences of \( S \). If \( \operatorname{rng} s_2 \subseteq \text{dom} \ h \) and \( s_3 \) is a subsequence of \( s_2 \), then \( h \cdot s_3 \) is a subsequence of \( h \cdot s_2 \).
9. Let \( h \) be a partial function from the carrier of \( S \) to \( \mathbb{R} \) and \( s_2, s_3 \) be sequences of \( S \). If \( \operatorname{rng} s_2 \subseteq \text{dom} \ h \) and \( s_3 \) is a subsequence of \( s_2 \), then \( h \cdot s_3 \) is a subsequence of \( h \cdot s_2 \).
10. \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:
    - (i) \( x_0 \in \text{dom} \ f \), and
    - (ii) for every \( r \) such that \( 0 < r \) there exists \( s \) such that \( 0 < s \) and for every \( x_1 \) such that \( x_1 \in \text{dom} \ f \) and \( \|x_1 - x_0\| < s \) holds \( \|f_{x_1} - f_{x_0}\| < r \).
11. Let \( f \) be a partial function from the carrier of \( S \) to \( \mathbb{R} \). Then \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:
    - (i) \( x_0 \in \text{dom} \ f \), and
    - (ii) for every \( r \) such that \( 0 < r \) there exists \( s \) such that \( 0 < s \) and for every \( x_1 \) such that \( x_1 \in \text{dom} \ f \) and \( \|x_1 - x_0\| < s \) holds \( |f_{x_1} - f_{x_0}| < r \).
12. Let given \( f, x_0 \). Then \( f \) is continuous in \( x_0 \) if and only if the following conditions are satisfied:
    - (i) \( x_0 \in \text{dom} \ f \), and
    - (ii) for every neighbourhood \( N_2 \) of \( f_{x_0} \) there exists a neighbourhood \( N \) of \( x_0 \) such that for every \( x_1 \) such that \( x_1 \in \text{dom} \ f \) and \( x_1 \in N \) holds \( f_{x_1} \in N_2 \).
(17) Let given $f$, $x_0$. Then $f$ is continuous in $x_0$ if and only if the following conditions are satisfied:
(i) $x_0 \in \text{dom } f$, and
(ii) for every neighbourhood $N_2$ of $f(x_0)$ there exists a neighbourhood $N$ of $x_0$ such that $f \circ N \subseteq N_2$.
(18) If $x_0 \in \text{dom } f$ and there exists a neighbourhood $N$ of $x_0$ such that $\text{dom } f \cap N = \{x_0\}$, then $f$ is continuous in $x_0$.
(19) Let $h_1$, $h_2$ be partial functions from $S$ to $T$ and $s_4$ be a sequence of $S$. If $\text{rng } s_4 \subseteq \text{dom } h_1 \cap \text{dom } h_2$, then $(h_1 + h_2) \cdot s_4 = h_1 \cdot s_4 + h_2 \cdot s_4$ and $(h_1 - h_2) \cdot s_4 = h_1 \cdot s_4 - h_2 \cdot s_4$.
(20) Let $h$ be a partial function from $S$ to $T$, $s_4$ be a sequence of $S$, and $r$ be a real number. If $\text{rng } s_4 \subseteq \text{dom } h$, then $(r \cdot h) \cdot s_4 = r \cdot (h \cdot s_4)$.
(21) Let $h$ be a partial function from $S$ to $T$ and $s_4$ be a sequence of $S$. If $\text{rng } s_4 \subseteq \text{dom } h$, then $\|h \cdot s_4\| = \|h\| \cdot \|s_4\|$ and $-h \cdot s_4 = (-h) \cdot s_4$.
(22) If $f_1$ is continuous in $x_0$ and $f_2$ is continuous in $x_0$, then $f_1 + f_2$ is continuous in $x_0$ and $f_1 - f_2$ is continuous in $x_0$.
(23) If $f$ is continuous in $x_0$, then $r \cdot f$ is continuous in $x_0$.
(24) If $f$ is continuous in $x_0$, then $\|f\|$ is continuous in $x_0$ and $-f$ is continuous in $x_0$.

Let us consider $S$, $T$ and let us consider $f$, $X$. We say that $f$ is continuous on $X$ if and only if:

(Def. 11) $X \subseteq \text{dom } f$ and for every $x_0 \in X$ holds $f \mid X$ is continuous in $x_0$.

Let us consider $S$, let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$, and let us consider $X$. We say that $f$ is continuous on $X$ if and only if:

(Def. 12) $X \subseteq \text{dom } f$ and for every $x_0 \in X$ holds $f \mid X$ is continuous in $x_0$.

One can prove the following propositions:
(25) Let given $X$, $f$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:
(i) $X \subseteq \text{dom } f$, and
(ii) for every $s_1$ such that $\text{rng } s_1 \subseteq X$ and $s_1$ is convergent and $\lim s_1 \in X$ holds $f \cdot s_1$ is convergent and $\lim s_1 = \lim (f \cdot s_1)$.
(26) $f$ is continuous on $X$ if and only if the following conditions are satisfied:
(i) $X \subseteq \text{dom } f$, and
(ii) for all $x_0$, $r$ such that $x_0 \in X$ and $0 < r$ there exists $s$ such that $0 < s$ and for every $x_1$ such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $\|f_{x_1} - f_{x_0}\| < r$.
(27) Let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$. Then $f$ is continuous on $X$ if and only if the following conditions are satisfied:
(i) $X \subseteq \text{dom } f$, and
(ii) for all $x_0$, $r$ such that $x_0 \in X$ and $0 < r$ there exists $s$ such that $0 < s$ and for every $x_1$ such that $x_1 \in X$ and $\|x_1 - x_0\| < s$ holds $|f(x_1) - f(x_0)| < r$.

(28) $f$ is continuous on $X$ if and only if $f|X$ is continuous on $X$.

(29) Let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$. Then $f$ is continuous on $X$ if and only if $f|X$ is continuous on $X$.

(30) If $f$ is continuous on $X$ and $X_1 \subseteq X$, then $f$ is continuous on $X_1$.

(31) If $x_0 \in \text{dom } f$, then $f$ is continuous on $\{x_0\}$.

(32) For all $X, f_1, f_2$ such that $f_1$ is continuous on $X$ and $f_2$ is continuous on $X$ holds $f_1 + f_2$ is continuous on $X$ and $f_1 - f_2$ is continuous on $X$.

(33) Let given $X, X_1, f_1, f_2$. Suppose $f_1$ is continuous on $X$ and $f_2$ is continuous on $X_1$. Then $f_1 + f_2$ is continuous on $X \cap X_1$ and $f_1 - f_2$ is continuous on $X \cap X_1$.

(34) For all $r, X, f$ such that $f$ is continuous on $X$ holds $rf$ is continuous on $X$.

(35) If $f$ is continuous on $X$, then $\|f\|$ is continuous on $X$ and $-f$ is continuous on $X$.

(36) Suppose $f$ is total and for all $x_1, x_2$ holds $f_{x_1 + x_2} = f_{x_1} + f_{x_2}$ and there exists $x_0$ such that $f$ is continuous in $x_0$. Then $f$ is continuous on the carrier of $S$.

(37) For every $f$ such that $\text{dom } f$ is compact and $f$ is continuous on $\text{dom } f$ holds $\text{rng } f$ is compact.

(38) Let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$. If $\text{dom } f$ is compact and $f$ is continuous on $\text{dom } f$, then $\text{rng } f$ is compact.

(39) If $Y \subseteq \text{dom } f$ and $Y$ is compact and $f$ is continuous on $Y$, then $f^\circ Y$ is compact.

(40) Let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$. Suppose $\text{dom } f \neq \emptyset$ and $\text{dom } f$ is compact and $f$ is continuous on $\text{dom } f$. Then there exist $x_1, x_2$ such that $x_1 \in \text{dom } f$ and $x_2 \in \text{dom } f$ and $f_{x_1} = \text{sup } \text{rng } f$ and $f_{x_2} = \text{inf } \text{rng } f$.

(41) Let given $f$. Suppose $\text{dom } f \neq \emptyset$ and $\text{dom } f$ is compact and $f$ is continuous on $\text{dom } f$. Then there exist $x_1, x_2$ such that $x_1 \in \text{dom } f$ and $x_2 \in \text{dom } f$ and $\|f\|_{x_1} = \text{sup } \text{rng } f$ and $\|f\|_{x_2} = \text{inf } \text{rng } f$.

(42) $\|f\|_{\text{X}} = \|f|_{\text{X}}\$.

(43) Let given $f, Y$. Suppose $Y \neq \emptyset$ and $Y \subseteq \text{dom } f$ and $Y$ is compact and $f$ is continuous on $Y$. Then there exist $x_1, x_2$ such that $x_1 \in Y$ and $x_2 \in Y$ and $\|f\|_{x_1} = \text{sup } (\|f\|_{\text{Y}})$ and $\|f\|_{x_2} = \text{inf } (\|f\|_{\text{Y}})$.

(44) Let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$ and given $Y$. Suppose $Y \neq \emptyset$ and $Y \subseteq \text{dom } f$ and $Y$ is compact and $f$ is continuous on $Y$. Then there exist $x_1, x_2$ such that $x_1 \in Y$ and $x_2 \in Y$ and $f_{x_1} = \text{sup } (f^\circ Y)$.
and $f_{x_2} = \inf(f \circ Y)$.

Let us consider $S$, $T$ and let us consider $X$, $f$. We say that $f$ is Lipschitzian on $X$ if and only if:

(Def. 13) $X \subseteq \text{dom } f$ and there exists $r$ such that $0 < r$ and for all $x_1$, $x_2$ such that $x_1 \in X$ and $x_2 \in X$ holds $\|f_{x_1} - f_{x_2}\| \leq r \cdot \|x_1 - x_2\|$.

Let us consider $S$, let us consider $X$, and let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$. We say that $f$ is Lipschitzian on $X$ if and only if:

(Def. 14) $X \subseteq \text{dom } f$ and there exists $r$ such that $0 < r$ and for all $x_1$, $x_2$ such that $x_1 \in X$ and $x_2 \in X$ holds $|f_{x_1} - f_{x_2}| \leq r \cdot \|x_1 - x_2\|$.

The following propositions are true:

(45) If $f$ is Lipschitzian on $X$ and $X_1 \subseteq X$, then $f$ is Lipschitzian on $X_1$.

(46) If $f_1$ is Lipschitzian on $X$ and $f_2$ is Lipschitzian on $X_1$, then $f_1 + f_2$ is Lipschitzian on $X \cap X_1$.

(47) If $f_1$ is Lipschitzian on $X$ and $f_2$ is Lipschitzian on $X_1$, then $f_1 - f_2$ is Lipschitzian on $X \cap X_1$.

(48) If $f$ is Lipschitzian on $X$, then $pf$ is Lipschitzian on $X$.

(49) If $f$ is Lipschitzian on $X$, then $-f$ is Lipschitzian on $X$ and $\|f\|$ is Lipschitzian on $X$.

(50) If $X \subseteq \text{dom } f$ and $f$ is a constant on $X$, then $f$ is Lipschitzian on $X$.

(51) $\text{id}_Y$ is Lipschitzian on $Y$.

(52) If $f$ is Lipschitzian on $X$, then $f$ is continuous on $X$.

(53) Let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$. If $f$ is Lipschitzian on $X$, then $f$ is continuous on $X$.

(54) For every $f$ such that there exists a point $r$ of $T$ such that $\text{rng } f = \{r\}$ holds $f$ is continuous on $\text{dom } f$.

(55) If $X \subseteq \text{dom } f$ and $f$ is a constant on $X$, then $f$ is continuous on $X$.

(56) For every partial function $f$ from $S$ to $T$ such that for every $x_0$ such that $x_0 \in \text{dom } f$ holds $f_{x_0} = x_0$ holds $f$ is continuous on $\text{dom } f$.

(57) For every partial function $f$ from $S$ to $T$ such that $f = \text{id}_{\text{dom } f}$ holds $f$ is continuous on $\text{dom } f$.

(58) For every partial function $f$ from $S$ to $T$ such that $Y \subseteq \text{dom } f$ and $f|Y = \text{id}_Y$ holds $f$ is continuous on $Y$.

(59) Let $f$ be a partial function from $S$ to $T$, $r$ be a real number, and $p$ be a point of $S$. Suppose $X \subseteq \text{dom } f$ and for every $x_0$ such that $x_0 \in X$ holds $f_{x_0} = r \cdot x_0 + p$. Then $f$ is continuous on $X$.

(60) Let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$. If for every $x_0$ such that $x_0 \in \text{dom } f$ holds $f_{x_0} = \|x_0\|$, then $f$ is continuous on $\text{dom } f$.

(61) Let $f$ be a partial function from the carrier of $S$ to $\mathbb{R}$. If $X \subseteq \text{dom } f$ and for every $x_0$ such that $x_0 \in X$ holds $f_{x_0} = \|x_0\|$, then $f$ is continuous on $X$. 
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