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On Some Points of a Simple Closed Curve[1]

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The notation and terminology used here are introduced in the following papers: [26], [28], [2], [13], [1], [29], [5], [18], [17], [3], [14], [24], [9], [23], [4], [25], [7], [10], [11], [12], [19], [20], [22], [21], [6], [8], [15], [16], and [27].

1. On the Subsets of $\mathcal{E}^2_{\rm T}$

For simplicity, we follow the rules: C denotes a simple closed curve, P denotes a subset of $\mathcal{E}^2_{\rm T}$, R denotes a non empty subset of $\mathcal{E}^2_{\rm T}$, p denotes a point of $\mathcal{E}^2_{\rm T}$, and i, j, k, m, n denote natural numbers.

One can prove the following propositions:

(1) For every point p of $\mathcal{E}^n_{\rm T}$ holds $\{p\}$ is Bounded.

(2) For all real numbers s_1, t and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{[s, t]; s$ ranges over real numbers: $s_1 < s\}$ holds P is convex.

(3) For all real numbers s_2, t and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{[s, t]; s$ ranges over real numbers: $s < s_2\}$ holds P is convex.

(4) For all real numbers s, t_1 and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{[s, t]; t$ ranges over real numbers: $t_1 < t\}$ holds P is convex.

(5) For all real numbers s, t_2 and for every subset P of $\mathcal{E}^2_{\rm T}$ such that $P = \{[s, t]; t$ ranges over real numbers: $t < t_2\}$ holds P is convex.

(6) NorthHalfline $p \setminus \{p\}$ is convex.

(7) SouthHalfline $p \setminus \{p\}$ is convex.

(8) WestHalfline $p \setminus \{p\}$ is convex.

(9) EastHalfline $p \setminus \{p\}$ is convex.

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(10) For every subset A of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{UBD}\,A$ misses A.

(11) Let P be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2 and $p_1 \neq q_1$ and $p_2 \neq q_2$. Then $p_1 \notin \mathrm{Segment}(P, p_1, p_2, q_1, q_2)$ and $p_2 \notin \mathrm{Segment}(P, p_1, p_2, q_1, q_2)$.

(12) $\mathrm{proj2}^\circ(C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}))$ is not empty.

(13) For every compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{proj2}^\circ(C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}))$ is closed, lower bounded, and upper bounded.

2. Gauges

The following propositions are true:

(14) $\langle 1,\ 1\rangle \in$ the indices of $\mathrm{Gauge}(R, n)$.

(15) $\langle 1,\ 2\rangle \in$ the indices of $\mathrm{Gauge}(R, n)$.

(16) $\langle 2,\ 1\rangle \in$ the indices of $\mathrm{Gauge}(R, n)$.

(17) Let C be a non vertical non horizontal compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $m > k$ and $\langle i,\ j\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$ and $\langle i,\ j+1\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$. Then $\rho(\mathrm{Gauge}(C, m) \circ (i, j), \mathrm{Gauge}(C, m) \circ (i, j+1)) < \rho(\mathrm{Gauge}(C, k) \circ (i, j), \mathrm{Gauge}(C, k) \circ (i, j+1))$.

(18) For every non vertical non horizontal compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $m > k$ holds $\rho(\mathrm{Gauge}(C, m) \circ (1, 1), \mathrm{Gauge}(C, m) \circ (1, 2)) < \rho(\mathrm{Gauge}(C, k) \circ (1, 1), \mathrm{Gauge}(C, k) \circ (1, 2))$.

(19) Let C be a non vertical non horizontal compact subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $m > k$ and $\langle i,\ j\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$ and $\langle i+1,\ j\rangle \in$ the indices of $\mathrm{Gauge}(C, k)$. Then $\rho(\mathrm{Gauge}(C, m) \circ (i, j), \mathrm{Gauge}(C, m) \circ (i+1, j)) < \rho(\mathrm{Gauge}(C, k) \circ (i, j), \mathrm{Gauge}(C, k) \circ (i+1, j))$.

(20) For every non vertical non horizontal compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ such that $m > k$ holds $\rho(\mathrm{Gauge}(C, m) \circ (1, 1), \mathrm{Gauge}(C, m) \circ (2, 1)) < \rho(\mathrm{Gauge}(C, k) \circ (1, 1), \mathrm{Gauge}(C, k) \circ (2, 1))$.

(21) Let r, t be real numbers. Suppose $r > 0$ and $t > 0$. Then there exists a natural number n such that $i < n$ and $\rho(\mathrm{Gauge}(C, n) \circ (1, 1), \mathrm{Gauge}(C, n) \circ (1, 2)) < r$ and $\rho(\mathrm{Gauge}(C, n) \circ (1, 1), \mathrm{Gauge}(C, n) \circ (2, 1)) < t$.

3. Middle Points

We now state four propositions:

(22) $\mathrm{UpperMiddlePoint}\,C \in C$.

(23) $\mathrm{LowerMiddlePoint}\,C \in C$.

(24) $(\mathrm{LowerMiddlePoint}\,C)_{\mathbf{2}} \neq (\mathrm{UpperMiddlePoint}\,C)_{\mathbf{2}}$.

(25) LowerMiddlePoint $C \neq$ UpperMiddlePoint C.

4. UpperArc and LowerArc

Next we state several propositions:

(26) W-bound(C) = W-bound(UpperArc(C)).

(27) E-bound(C) = E-bound(UpperArc(C)).

(28) W-bound(C) = W-bound(LowerArc(C)).

(29) E-bound(C) = E-bound(LowerArc(C)).

(30) UpperArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ is not empty and proj2°(UpperArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$) is not empty.

(31) LowerArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ is not empty and proj2°(LowerArc$(C) \cap$ VerticalLine$(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$) is not empty.

(32) For every compact connected subset P of $\mathcal{E}^2_{\rm T}$ such that $P \subseteq C$ and $\mathrm{W}_{\min}(C) \in P$ and $\mathrm{E}_{\max}(C) \in P$ holds UpperArc$(C) \subseteq P$ or LowerArc$(C) \subseteq P$.

5. UMP and LMP

Let P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. The functor UMP P yielding a point of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 1) UMP $P = [\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2}$, sup(proj2°($P \cap$ VerticalLine$(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})$))].

The functor LMP P yielding a point of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 2) LMP $P = [\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2}$, inf(proj2°($P \cap$ VerticalLine$(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})$))].

We now state a number of propositions:

(33) $(\text{UMP}\,P)_{\mathbf{1}} = \frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}$.

(34) $(\text{UMP}\,P)_{\mathbf{2}}$ = sup(proj2°($P \cap$ VerticalLine$(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})$)).

(35) $(\text{LMP}\,P)_{\mathbf{1}} = \frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}$.

(36) $(\text{LMP}\,P)_{\mathbf{2}}$ = inf(proj2°($P \cap$ VerticalLine$(\frac{\text{E-bound}(P)+\text{W-bound}(P)}{2})$)).

(37) For every non vertical compact subset C of $\mathcal{E}^2_{\rm T}$ holds UMP $C \neq \mathrm{W}_{\min}(C)$.

(38) For every non vertical compact subset C of $\mathcal{E}^2_{\rm T}$ holds UMP $C \neq \mathrm{E}_{\max}(C)$.

(39) For every non vertical compact subset C of $\mathcal{E}^2_{\rm T}$ holds LMP $C \neq \mathrm{W}_{\min}(C)$.

(40) For every non vertical compact subset C of $\mathcal{E}^2_{\rm T}$ holds LMP $C \neq \mathrm{E}_{\max}(C)$.

(41) For every compact subset C of $\mathcal{E}^2_{\rm T}$ such that $p \in C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ holds $p_{\mathbf{2}} \leq (\mathrm{UMP}\, C)_{\mathbf{2}}$.

(42) For every compact subset C of $\mathcal{E}^2_{\rm T}$ such that $p \in C \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2})$ holds $(\mathrm{LMP}\, C)_{\mathbf{2}} \leq p_{\mathbf{2}}$.

(43) $\mathrm{UMP}\, C \in C$.

(44) $\mathrm{LMP}\, C \in C$.

(45) $\mathcal{L}(\mathrm{UMP}\, P, [\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}, \text{N-bound}(P)])$ is vertical.

(46) $\mathcal{L}(\mathrm{LMP}\, P, [\frac{\text{W-bound}(P)+\text{E-bound}(P)}{2}, \text{S-bound}(P)])$ is vertical.

(47) $\mathcal{L}(\mathrm{UMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{N-bound}(C)]) \cap C = \{\mathrm{UMP}\, C\}$.

(48) $\mathcal{L}(\mathrm{LMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{S-bound}(C)]) \cap C = \{\mathrm{LMP}\, C\}$.

(49) $(\mathrm{LMP}\, C)_{\mathbf{2}} < (\mathrm{UMP}\, C)_{\mathbf{2}}$.

(50) $\mathrm{UMP}\, C \neq \mathrm{LMP}\, C$.

(51) $\text{S-bound}(C) < (\mathrm{UMP}\, C)_{\mathbf{2}}$.

(52) $(\mathrm{UMP}\, C)_{\mathbf{2}} \leq \text{N-bound}(C)$.

(53) $\text{S-bound}(C) \leq (\mathrm{LMP}\, C)_{\mathbf{2}}$.

(54) $(\mathrm{LMP}\, C)_{\mathbf{2}} < \text{N-bound}(C)$.

(55) $\mathcal{L}(\mathrm{UMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{N-bound}(C)])$ misses $\mathcal{L}(\mathrm{LMP}\, C, [\frac{\text{W-bound}(C)+\text{E-bound}(C)}{2}, \text{S-bound}(C)])$.

(56) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $A \subseteq B$ and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^\circ(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2}))$ is upper bounded. Then $(\mathrm{UMP}\, A)_{\mathbf{2}} \leq (\mathrm{UMP}\, B)_{\mathbf{2}}$.

(57) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $A \subseteq B$ and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^\circ(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2}))$ is lower bounded. Then $(\mathrm{LMP}\, B)_{\mathbf{2}} \leq (\mathrm{LMP}\, A)_{\mathbf{2}}$.

(58) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $A \subseteq B$ and $\mathrm{UMP}\, B \in A$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^\circ(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(B)+\text{E-bound}(B)}{2}))$ is upper bounded and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$. Then $\mathrm{UMP}\, A = \mathrm{UMP}\, B$.

(59) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$. Suppose $A \subseteq B$ and $\mathrm{LMP}\, B \in A$ and $A \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(A)+\text{E-bound}(A)}{2})$ is non empty and $\mathrm{proj2}^\circ(B \cap \mathrm{VerticalLine}(\frac{\text{W-bound}(B)+\text{E-bound}(B)}{2}))$ is lower bounded and $\text{W-bound}(A) + \text{E-bound}(A) = \text{W-bound}(B) + \text{E-bound}(B)$. Then $\mathrm{LMP}\, A = \mathrm{LMP}\, B$.

(60) $(\mathrm{UMP}\,\mathrm{UpperArc}(C))_{\mathbf{2}} \leq \text{N-bound}(C)$.

(61) $\text{S-bound}(C) \leq (\mathrm{LMP}\,\mathrm{LowerArc}(C))_{\mathbf{2}}$.

(62) $\mathrm{LMP}\,C \notin \mathrm{LowerArc}(C)$ or $\mathrm{UMP}\,C \notin \mathrm{LowerArc}(C)$.

(63) $\mathrm{LMP}\,C \notin \mathrm{UpperArc}(C)$ or $\mathrm{UMP}\,C \notin \mathrm{UpperArc}(C)$.

(64) If $0 < n$, then $\sup(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), \mathrm{len}\,\mathrm{Gauge}(C,n))))) = \sup(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathrm{VerticalLine}(\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))+\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2})))$.

(65) If $0 < n$, then $\inf(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), \mathrm{len}\,\mathrm{Gauge}(C,n))))) = \inf(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathrm{VerticalLine}(\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))+\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2})))$.

(66) If $0 < n$, then $\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = [\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))+\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2}, \sup(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), \mathrm{len}\,\mathrm{Gauge}(C,n)))))]$.

(67) If $0 < n$, then $\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = [\frac{\text{E-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))+\text{W-bound}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))}{2}, \inf(\mathrm{proj2}^{\circ}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) \cap \mathcal{L}(\mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n),1), \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), \mathrm{len}\,\mathrm{Gauge}(C,n)))))]$.

(68) $(\mathrm{UMP}\,C)_{\mathbf{2}} < (\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))_{\mathbf{2}}$.

(69) $(\mathrm{LMP}\,C)_{\mathbf{2}} > (\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))_{\mathbf{2}}$.

(70) $\mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \in \mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(71) $\mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))) \in \mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(72) If $0 < n$, then there exists a natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), i)$.

(73) If $0 < n$, then there exists a natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{Gauge}(C,n) \circ (\mathrm{Center}\,\mathrm{Gauge}(C,n), i)$.

(74) If $0 < n$, then $\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(75) If $0 < n$, then $\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)) = \mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n)))$.

(76) If $0 < n$, then $(\mathrm{UMP}\,C)_{\mathbf{2}} < (\mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))))_{\mathbf{2}}$.

(77) If $0 < n$, then $(\mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))))_{\mathbf{2}} < (\mathrm{LMP}\,C)_{\mathbf{2}}$.

(78) If $i \leq j$, then $(\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)))_{\mathbf{2}} \leq (\mathrm{UMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))_{\mathbf{2}}$.

(79) If $i \leq j$, then $(\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i)))_{\mathbf{2}} \leq (\mathrm{LMP}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j)))_{\mathbf{2}}$.

(80) If $0 < i$ and $i \leq j$, then $(\mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j))))_{\mathbf{2}} \leq (\mathrm{UMP}\,\mathrm{UpperArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i))))_{\mathbf{2}}$.

(81) If $0 < i$ and $i \leq j$, then $(\mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,i))))_{\mathbf{2}} \leq (\mathrm{LMP}\,\mathrm{LowerArc}(\widetilde{\mathcal{L}}(\mathrm{Cage}(C,j))))_{\mathbf{2}}$.

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References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Gauges. *Formalized Mathematics*, 8(**1**):25–27, 1999.
[7] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.
[8] Czesław Byliński and Mariusz Żynel. Cages – the external approximation of Jordan's curve. *Formalized Mathematics*, 9(**1**):19–24, 2001.
[9] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[10] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.
[12] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Simple closed curves. *Formalized Mathematics*, 2(**5**):663–664, 1991.
[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[14] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.
[15] Artur Korniłowicz. Properties of left and right components. *Formalized Mathematics*, 8(**1**):163–168, 1999.
[16] Artur Korniłowicz, Robert Milewski, Adam Naumowicz, and Andrzej Trybulec. Gauges and cages. Part I. *Formalized Mathematics*, 9(**3**):501–509, 2001.
[17] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[18] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[19] Yatsuka Nakamura and Czesław Byliński. Extremal properties of vertices on special polygons. Part I. *Formalized Mathematics*, 5(**1**):97–102, 1996.
[20] Yatsuka Nakamura and Jarosław Kotowicz. The Jordan's property for certain subsets of the plane. *Formalized Mathematics*, 3(**2**):137–142, 1992.
[21] Yatsuka Nakamura and Andrzej Trybulec. A decomposition of a simple closed curves and the order of their points. *Formalized Mathematics*, 6(**4**):563–572, 1997.
[22] Yatsuka Nakamura, Andrzej Trybulec, and Czesław Byliński. Bounded domains and unbounded domains. *Formalized Mathematics*, 8(**1**):1–13, 1999.
[23] Beata Padlewska. Connected spaces. *Formalized Mathematics*, 1(**1**):239–244, 1990.
[24] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[25] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[26] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[27] Andrzej Trybulec. On the minimal distance between sets in Euclidean space. *Formalized Mathematics*, 10(**3**):153–158, 2002.
[28] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[29] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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