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Inverse Trigonometric Functions Arctan and Arccot

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Summary. This article describes definitions of inverse trigonometric functions arctan, arccot and their main properties, as well as several differentiation formulas of arctan and arccot.

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The articles [17], [1], [2], [18], [3], [13], [19], [7], [15], [5], [9], [12], [16], [4], [6], [8], [11], [14], and [10] provide the notation and terminology for this paper.

1. Function Arctan and Arccot

For simplicity, we adopt the following convention: x, r, s, h denote real numbers, n denotes an element of $\mathbb{N}$, Z denotes an open subset of $\mathbb{R}$, and f, f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathbb{R}$.

The following propositions are true:

(1) $]-\frac{\pi}{2}, \frac{\pi}{2}[\subseteq \mathrm{dom}$ (the function tan).

(2) $]0, \pi[\subseteq \mathrm{dom}$ (the function cot).

(3)(i) The function tan is differentiable on $]-\frac{\pi}{2}, \frac{\pi}{2}[$, and

(ii) for every x such that $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ holds (the function tan)$'(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$.

(4) The function cot is differentiable on $]0, \pi[$ and for every x such that $x \in]0, \pi[$ holds (the function cot)$'(x) = -\frac{1}{(\sin x)^{\mathbf{2}}}$.

(5) The function tan is continuous on $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(6) The function cot is continuous on $]0, \pi[$.

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(7) The function tan is increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(8) The function cot is decreasing on $]0, \pi[$.

(9) (The function tan)$\restriction]-\frac{\pi}{2}, \frac{\pi}{2}[$ is one-to-one.

(10) (The function cot)$\restriction]0, \pi[$ is one-to-one.

Let us mention that (the function tan)$\restriction]-\frac{\pi}{2}, \frac{\pi}{2}[$ is one-to-one and (the function cot)$\restriction]0, \pi[$ is one-to-one.

The partial function the function arctan from $\mathbb{R}$ to $\mathbb{R}$ is defined as follows:

(Def. 1) The function arctan = ((the function tan)$\restriction]-\frac{\pi}{2}, \frac{\pi}{2}[)^{-1}$.

The partial function the function arccot from $\mathbb{R}$ to $\mathbb{R}$ is defined by:

(Def. 2) The function arccot = ((the function cot)$\restriction]0, \pi[)^{-1}$.

Let r be a real number. The functor $\operatorname{arctan} r$ is defined by:

(Def. 3) $\operatorname{arctan} r$ = (the function arctan)(r).

The functor $\operatorname{arccot} r$ is defined by:

(Def. 4) $\operatorname{arccot} r$ = (the function arccot)(r).

Let r be a real number. Then $\operatorname{arctan} r$ is a real number. Then $\operatorname{arccot} r$ is a real number.

We now state two propositions:

(11) rng (the function arctan) $=]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(12) rng (the function arccot) $=]0, \pi[$.

Let us mention that the function arctan is one-to-one and the function arccot is one-to-one.

Let r be a real number. Then $\tan r$ is a real number. Then $\cot r$ is a real number.

Next we state a number of propositions:

(13) For every real number x such that $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ holds (the function tan)(x) $= \tan x$.

(14) For every real number x such that $x \in]0, \pi[$ holds (the function cot)(x) $= \cot x$.

(15) For every real number x such that $\cos x \neq 0$ holds (the function tan)(x) $= \tan x$.

(16) For every real number x such that (the function sin)(x) $\neq 0$ holds (the function cot)(x) $= \cot x$.

(17) $\tan(-\frac{\pi}{4}) = -1$.

(18) $\cot(\frac{\pi}{4}) = 1$ and $\cot(\frac{3}{4} \cdot \pi) = -1$.

(19) For every real number x such that $x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ holds $\tan x \in [-1, 1]$.

(20) For every real number x such that $x \in [\frac{\pi}{4}, \frac{3}{4} \cdot \pi]$ holds $\cot x \in [-1, 1]$.

(21) rng((the function tan)$\restriction[-\frac{\pi}{4}, \frac{\pi}{4}]) = [-1, 1]$.

(22) rng((the function cot)$\restriction[\frac{\pi}{4}, \frac{3}{4} \cdot \pi]) = [-1, 1]$.

(23) $[-1,1] \subseteq \operatorname{dom}$ (the function arctan).

(24) $[-1,1] \subseteq \operatorname{dom}$ (the function arccot).

Let us observe that (the function tan)$\restriction[-\frac{\pi}{4}, \frac{\pi}{4}]$ is one-to-one and (the function cot)$\restriction[\frac{\pi}{4}, \frac{3}{4} \cdot \pi]$ is one-to-one.

The following propositions are true:

(25) (The function arctan)$\restriction[-1,1] =$ ((the function tan)$\restriction[-\frac{\pi}{4}, \frac{\pi}{4}])^{-1}$.

(26) (The function arccot)$\restriction[-1,1] =$ ((the function cot)$\restriction[\frac{\pi}{4}, \frac{3}{4} \cdot \pi])^{-1}$.

(27) ((The function tan)$\restriction[-\frac{\pi}{4}, \frac{\pi}{4}]$ **qua** function) $\cdot$((the function arctan)$\restriction[-1,1]) = \mathrm{id}_{[-1,1]}$.

(28) ((The function cot)$\restriction[\frac{\pi}{4}, \frac{3}{4} \cdot \pi]$ **qua** function) $\cdot$((the function arccot)$\restriction[-1,1]) = \mathrm{id}_{[-1,1]}$.

(29) ((The function tan)$\restriction[-\frac{\pi}{4}, \frac{\pi}{4}]) \cdot$ ((the function arctan)$\restriction[-1,1]) = \mathrm{id}_{[-1,1]}$.

(30) ((The function cot)$\restriction[\frac{\pi}{4}, \frac{3}{4} \cdot \pi]) \cdot$ ((the function arccot)$\restriction[-1,1]) = \mathrm{id}_{[-1,1]}$.

(31) (The function arctan **qua** function) $\cdot$((the function tan)$\restriction]-\frac{\pi}{2}, \frac{\pi}{2}[) = \mathrm{id}_{]-\frac{\pi}{2}, \frac{\pi}{2}[}$.

(32) (The function arccot) $\cdot$((the function cot)$\restriction]0, \pi[) = \mathrm{id}_{]0,\pi[}$.

(33) (The function arctan **qua** function) $\cdot$((the function tan)$\restriction]-\frac{\pi}{2}, \frac{\pi}{2}[) = \mathrm{id}_{]-\frac{\pi}{2}, \frac{\pi}{2}[}$.

(34) (The function arccot **qua** function) $\cdot$((the function cot)$\restriction]0, \pi[) = \mathrm{id}_{]0,\pi[}$.

(35) If $-\frac{\pi}{2} < r < \frac{\pi}{2}$, then $\arctan \tan r = r$.

(36) If $0 < r < \pi$, then $\operatorname{arccot} \cot r = r$.

(37) $\arctan(-1) = -\frac{\pi}{4}$.

(38) $\operatorname{arccot}(-1) = \frac{3}{4} \cdot \pi$.

(39) $\arctan 1 = \frac{\pi}{4}$.

(40) $\operatorname{arccot} 1 = \frac{\pi}{4}$.

(41) $\tan 0 = 0$.

(42) $\cot(\frac{\pi}{2}) = 0$.

(43) $\arctan 0 = 0$.

(44) $\operatorname{arccot} 0 = \frac{\pi}{2}$.

(45) The function arctan is increasing on (the function tan) $^\circ]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(46) The function arccot is decreasing on (the function cot) $^\circ]0, \pi[$.

(47) The function arctan is increasing on $[-1,1]$.

(48) The function arccot is decreasing on $[-1,1]$.

(49) For every real number x such that $x \in [-1,1]$ holds $\arctan x \in [-\frac{\pi}{4}, \frac{\pi}{4}]$.

(50) For every real number x such that $x \in [-1,1]$ holds $\operatorname{arccot} x \in [\frac{\pi}{4}, \frac{3}{4} \cdot \pi]$.

(51) If $-1 \leq r \leq 1$, then $\tan \arctan r = r$.

(52) If $-1 \leq r \leq 1$, then $\cot \operatorname{arccot} r = r$.

(53) The function arctan is continuous on $[-1,1]$.

(54) The function arccot is continuous on $[-1,1]$.

(55) rng((the function arctan)$\restriction[-1,1]) = [-\frac{\pi}{4}, \frac{\pi}{4}]$.

(56) rng((the function arccot)$\restriction[-1,1]) = [\frac{\pi}{4}, \frac{3}{4} \cdot \pi]$.

(57) If $-1 \leq r \leq 1$ and $\arctan r = -\frac{\pi}{4}$, then $r = -1$.

(58) If $-1 \leq r \leq 1$ and $\operatorname{arccot} r = \frac{3}{4} \cdot \pi$, then $r = -1$.

(59) If $-1 \leq r \leq 1$ and $\arctan r = 0$, then $r = 0$.

(60) If $-1 \leq r \leq 1$ and $\operatorname{arccot} r = \frac{\pi}{2}$, then $r = 0$.

(61) If $-1 \leq r \leq 1$ and $\arctan r = \frac{\pi}{4}$, then $r = 1$.

(62) If $-1 \leq r \leq 1$ and $\operatorname{arccot} r = \frac{\pi}{4}$, then $r = 1$.

(63) If $-1 \leq r \leq 1$, then $-\frac{\pi}{4} \leq \arctan r \leq \frac{\pi}{4}$.

(64) If $-1 \leq r \leq 1$, then $\frac{\pi}{4} \leq \operatorname{arccot} r \leq \frac{3}{4} \cdot \pi$.

(65) If $-1 < r < 1$, then $-\frac{\pi}{4} < \arctan r < \frac{\pi}{4}$.

(66) If $-1 < r < 1$, then $\frac{\pi}{4} < \operatorname{arccot} r < \frac{3}{4} \cdot \pi$.

(67) If $-1 \leq r \leq 1$, then $\arctan r = -\arctan(-r)$.

(68) If $-1 \leq r \leq 1$, then $\operatorname{arccot} r = \pi - \operatorname{arccot}(-r)$.

(69) If $-1 \leq r \leq 1$, then $\cot \arctan r = \frac{1}{r}$.

(70) If $-1 \leq r \leq 1$, then $\tan \operatorname{arccot} r = \frac{1}{r}$.

(71) The function arctan is differentiable on (the function tan) $^\circ]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(72) The function arccot is differentiable on (the function cot) $^\circ]0, \pi[$.

(73) The function arctan is differentiable on $]-1,1[$.

(74) The function arccot is differentiable on $]-1,1[$.

(75) If $-1 \leq r \leq 1$, then (the function arctan)$'(r) = \frac{1}{1+r^{\mathbf{2}}}$.

(76) If $-1 \leq r \leq 1$, then (the function arccot)$'(r) = -\frac{1}{1+r^{\mathbf{2}}}$.

(77) The function arctan is continuous on (the function tan) $^\circ]-\frac{\pi}{2}, \frac{\pi}{2}[$.

(78) The function arccot is continuous on (the function cot) $^\circ]0, \pi[$.

(79) dom (the function arctan) is open.

(80) dom (the function arccot) is open.

2. Several Differentiation Formulas of Arctan and Arccot

We now state a number of propositions:

(81) Suppose $Z \subseteq]-1,1[$. Then the function arctan is differentiable on Z and for every x such that $x \in Z$ holds (the function arctan)$'_{\restriction Z}(x) = \frac{1}{1+x^{\mathbf{2}}}$.

(82) Suppose $Z \subseteq]-1,1[$. Then the function arccot is differentiable on Z and for every x such that $x \in Z$ holds (the function arccot)$'_{\restriction Z}(x) = -\frac{1}{1+x^{\mathbf{2}}}$.

(83) Suppose $Z \subseteq]-1,1[$. Then

(i) r the function arctan is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(r \text{ the function arctan})'_{\restriction Z}(x) = \frac{r}{1+x^{\mathbf{2}}}$.

(84) Suppose $Z \subseteq]-1,1[$. Then

(i) r the function arccot is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(r \text{ the function arccot})'_{\restriction Z}(x) = -\frac{r}{1+x^{\mathbf{2}}}$.

(85) Suppose f is differentiable in x and $-1 < f(x) < 1$. Then (the function arctan) $\cdot f$ is differentiable in x and $((\text{the function arctan}) \cdot f)'(x) = \frac{f'(x)}{1+f(x)^{\mathbf{2}}}$.

(86) Suppose f is differentiable in x and $-1 < f(x) < 1$. Then (the function arccot) $\cdot f$ is differentiable in x and $((\text{the function arccot}) \cdot f)'(x) = -\frac{f'(x)}{1+f(x)^{\mathbf{2}}}$.

(87) Suppose $Z \subseteq \text{dom}((\text{the function arctan}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + s$ and $-1 < f(x) < 1$. Then

(i) (the function arctan) $\cdot f$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function arctan}) \cdot f)'_{\restriction Z}(x) = \frac{r}{1+(r \cdot x+s)^{\mathbf{2}}}$.

(88) Suppose $Z \subseteq \text{dom}((\text{the function arccot}) \cdot f)$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + s$ and $-1 < f(x) < 1$. Then

(i) (the function arccot) $\cdot f$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function arccot}) \cdot f)'_{\restriction Z}(x) = -\frac{r}{1+(r \cdot x+s)^{\mathbf{2}}}$.

(89) Suppose $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function arctan}))$ and $Z \subseteq]-1,1[$ and for every x such that $x \in Z$ holds $\arctan x > 0$. Then

(i) (the function ln) $\cdot$(the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function ln}) \cdot (\text{the function arctan}))'_{\restriction Z}(x) = \frac{1}{(1+x^{\mathbf{2}}) \cdot \arctan x}$.

(90) Suppose $Z \subseteq \text{dom}((\text{the function ln}) \cdot (\text{the function arccot}))$ and $Z \subseteq]-1,1[$ and for every x such that $x \in Z$ holds $\text{arccot}\, x > 0$. Then

(i) (the function ln) $\cdot$(the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function ln}) \cdot (\text{the function arccot}))'_{\restriction Z}(x) = -\frac{1}{(1+x^{\mathbf{2}}) \cdot \text{arccot}\, x}$.

(91) Suppose $Z \subseteq \text{dom}((\square^{n}) \cdot \text{the function arctan})$ and $Z \subseteq]-1,1[$. Then

(i) $(\square^{n}) \cdot$ the function arctan is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\square^{n}) \cdot \text{the function arctan})'_{\restriction Z}(x) = \frac{n \cdot (\arctan x)^{n-1}}{1+x^{\mathbf{2}}}$.

(92) Suppose $Z \subseteq \text{dom}((\square^{n}) \cdot \text{the function arccot})$ and $Z \subseteq]-1,1[$. Then

(i) $(\square^{n}) \cdot$ the function arccot is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\square^n) \cdot \text{the function arccot})'_{\restriction Z}(x) = -\frac{n \cdot (\text{arccot}\, x)^{n-1}}{1+x^{\mathbf{2}}}$.

(93) Suppose $Z \subseteq \text{dom}(\frac{1}{2}\,((\square^2) \cdot \text{the function arctan}))$ and $Z \subseteq]-1, 1[$. Then

(i) $\frac{1}{2}\,((\square^2) \cdot \text{the function arctan})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{2}\,((\square^2) \cdot \text{the function arctan}))'_{\restriction Z}(x) = \frac{\arctan x}{1+x^{\mathbf{2}}}$.

(94) Suppose $Z \subseteq \text{dom}(\frac{1}{2}\,((\square^2) \cdot \text{the function arccot}))$ and $Z \subseteq]-1, 1[$. Then

(i) $\frac{1}{2}\,((\square^2) \cdot \text{the function arccot})$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{2}\,((\square^2) \cdot \text{the function arccot}))'_{\restriction Z}(x) = -\frac{\text{arccot}\, x}{1+x^{\mathbf{2}}}$.

(95) Suppose $Z \subseteq]-1, 1[$. Then

(i) id_Z the function arctan is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\text{id}_Z \text{ the function arctan})'_{\restriction Z}(x) = \arctan x + \frac{x}{1+x^{\mathbf{2}}}$.

(96) Suppose $Z \subseteq]-1, 1[$. Then

(i) id_Z the function arccot is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\text{id}_Z \text{ the function arccot})'_{\restriction Z}(x) = \text{arccot}\, x - \frac{x}{1+x^{\mathbf{2}}}$.

(97) Suppose $Z \subseteq \text{dom}(f \text{ the function arctan})$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + s$. Then

(i) f the function arctan is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(f \text{ the function arctan})'_{\restriction Z}(x) = r \cdot \arctan x + \frac{r \cdot x + s}{1+x^{\mathbf{2}}}$.

(98) Suppose $Z \subseteq \text{dom}(f \text{ the function arccot})$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x + s$. Then

(i) f the function arccot is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(f \text{ the function arccot})'_{\restriction Z}(x) = r \cdot \text{arccot}\, x - \frac{r \cdot x + s}{1+x^{\mathbf{2}}}$.

(99) Suppose $Z \subseteq \text{dom}(\frac{1}{2}\,((\text{the function arctan}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = 2 \cdot x$ and $-1 < f(x) < 1$. Then

(i) $\frac{1}{2}\,((\text{the function arctan}) \cdot f)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{2}\,((\text{the function arctan}) \cdot f))'_{\restriction Z}(x) = \frac{1}{1+(2 \cdot x)^{\mathbf{2}}}$.

(100) Suppose $Z \subseteq \text{dom}(\frac{1}{2}\,((\text{the function arccot}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = 2 \cdot x$ and $-1 < f(x) < 1$. Then

(i) $\frac{1}{2}\,((\text{the function arccot}) \cdot f)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{2}\,((\text{the function arccot}) \cdot f))'_{\restriction Z}(x) = -\frac{1}{1+(2 \cdot x)^{\mathbf{2}}}$.

(101) Suppose $Z \subseteq \text{dom}(f_1 + f_2)$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f_2 = \square^2$. Then $f_1 + f_2$ is differentiable on Z and for every

x such that $x \in Z$ holds $(f_1 + f_2)'_{\restriction Z}(x) = 2 \cdot x$.

(102) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2)))$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$. Then

(i) $\frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2))$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2)))'_{\restriction Z}(x) = \frac{x}{1+x^2}$.

(103) Suppose that

(i) $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\,\text{the function arctan} - \frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2)))$,

(ii) $Z \subseteq]-1, 1[$,

(iii) $f_2 = \square^2$, and

(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$.

Then

(v) $\mathrm{id}_Z\,\text{the function arctan} - \frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2))$ is differentiable on Z, and

(vi) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\,\text{the function arctan} - \frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2)))'_{\restriction Z}(x) = \arctan x$.

(104) Suppose that

(i) $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\,\text{the function arccot} + \frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2)))$,

(ii) $Z \subseteq]-1, 1[$,

(iii) $f_2 = \square^2$, and

(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$.

Then

(v) $\mathrm{id}_Z\,\text{the function arccot} + \frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2))$ is differentiable on Z, and

(vi) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\,\text{the function arccot} + \frac{1}{2}\,((\text{the function ln}) \cdot (f_1 + f_2)))'_{\restriction Z}(x) = \mathrm{arccot}\,x$.

(105) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\,((\text{the function arctan}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$ and $-1 < f(x) < 1$. Then

(i) $\mathrm{id}_Z\,((\text{the function arctan}) \cdot f)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\,((\text{the function arctan}) \cdot f))'_{\restriction Z}(x) = \arctan(\frac{x}{r}) + \frac{x}{r \cdot (1 + (\frac{x}{r})^2)}$.

(106) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\,((\text{the function arccot}) \cdot f))$ and for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$ and $-1 < f(x) < 1$. Then

(i) $\mathrm{id}_Z\,((\text{the function arccot}) \cdot f)$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\,((\text{the function arccot}) \cdot f))'_{\restriction Z}(x) = \mathrm{arccot}(\frac{x}{r}) - \frac{x}{r \cdot (1 + (\frac{x}{r})^2)}$.

(107) Suppose $Z \subseteq \mathrm{dom}(f_1 + f_2)$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f_2 = (\square^2) \cdot f$ and for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$. Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + f_2)'_{\restriction Z}(x) = \frac{2 \cdot x}{r^2}$.

(108) Suppose that

(i) $Z \subseteq \mathrm{dom}(\frac{r}{2}\,((\text{the function ln})\ \cdot(f_1 + f_2)))$,

(ii) for every x such that $x \in Z$ holds $f_1(x) = 1$,

(iii) $r \neq 0$,

(iv) $f_2 = (\square^2) \cdot f$, and

(v) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$.

Then

(vi) $\frac{r}{2}\,((\text{the function ln})\ \cdot(f_1 + f_2))$ is differentiable on Z, and

(vii) for every x such that $x \in Z$ holds $(\frac{r}{2}\,((\text{the function ln})\ \cdot(f_1 + f_2)))'_{\restriction Z}(x) = \frac{x}{r \cdot (1 + (\frac{x}{r})^{\mathbf{2}})}$.

(109) Suppose that

(i) $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\,((\text{the function arctan})\ \cdot f) - \frac{r}{2}\,((\text{the function ln})\ \cdot(f_1+f_2)))$,

(ii) $r \neq 0$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$ and $-1 < f(x) < 1$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,

(v) $f_2 = (\square^2) \cdot f$, and

(vi) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$.

Then

(vii) $\mathrm{id}_Z\,((\text{the function arctan})\ \cdot f) - \frac{r}{2}\,((\text{the function ln})\ \cdot(f_1 + f_2))$ is differentiable on Z, and

(viii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\,((\text{the function arctan})\ \cdot f) - \frac{r}{2}\,((\text{the function ln})\ \cdot(f_1 + f_2)))'_{\restriction Z}(x) = \arctan(\frac{x}{r})$.

(110) Suppose that

(i) $Z \subseteq \mathrm{dom}(\mathrm{id}_Z\,((\text{the function arccot})\ \cdot f) + \frac{r}{2}\,((\text{the function ln})\ \cdot(f_1+f_2)))$,

(ii) $r \neq 0$,

(iii) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$ and $-1 < f(x) < 1$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,

(v) $f_2 = (\square^2) \cdot f$, and

(vi) for every x such that $x \in Z$ holds $f(x) = \frac{x}{r}$.

Then

(vii) $\mathrm{id}_Z\,((\text{the function arccot})\ \cdot f) + \frac{r}{2}\,((\text{the function ln})\ \cdot(f_1 + f_2))$ is differentiable on Z, and

(viii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z\,((\text{the function arccot})\ \cdot f) + \frac{r}{2}\,((\text{the function ln})\ \cdot(f_1 + f_2)))'_{\restriction Z}(x) = \mathrm{arccot}(\frac{x}{r})$.

(111) Suppose $Z \subseteq \mathrm{dom}((\text{the function arctan})\ \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $-1 < (\frac{1}{f})(x) < 1$. Then

(i) $(\text{the function arctan})\ \cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function arctan})\ \cdot \frac{1}{f})'_{\restriction Z}(x) = -\frac{1}{1+x^{\mathbf{2}}}$.

(112) Suppose $Z \subseteq \mathrm{dom}((\text{the function arccot}) \cdot \frac{1}{f})$ and for every x such that $x \in Z$ holds $f(x) = x$ and $-1 < (\frac{1}{f})(x) < 1$. Then

(i) (the function arccot) $\cdot \frac{1}{f}$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function arccot}) \cdot \frac{1}{f})'_{\restriction Z}(x) = \frac{1}{1+x^{\mathbf{2}}}$.

(113) Suppose that

(i) $Z \subseteq \mathrm{dom}((\text{the function arctan}) \cdot f)$,

(ii) $f = f_1 + h\, f_2$,

(iii) for every x such that $x \in Z$ holds $-1 < f(x) < 1$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = r + s \cdot x$, and

(v) $f_2 = \square^2$.

Then

(vi) (the function arctan) $\cdot(f_1 + h\, f_2)$ is differentiable on Z, and

(vii) for every x such that $x \in Z$ holds $((\text{the function arctan}) \cdot(f_1 + h\, f_2))'_{\restriction Z}(x) = \frac{s+2 \cdot h \cdot x}{1+(r+s \cdot x+h \cdot x^{\mathbf{2}})^{\mathbf{2}}}$.

(114) Suppose that

(i) $Z \subseteq \mathrm{dom}((\text{the function arccot}) \cdot f)$,

(ii) $f = f_1 + h\, f_2$,

(iii) for every x such that $x \in Z$ holds $-1 < f(x) < 1$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = r + s \cdot x$, and

(v) $f_2 = \square^2$.

Then

(vi) (the function arccot) $\cdot(f_1 + h\, f_2)$ is differentiable on Z, and

(vii) for every x such that $x \in Z$ holds $((\text{the function arccot}) \cdot(f_1 + h\, f_2))'_{\restriction Z}(x) = -\frac{s+2 \cdot h \cdot x}{1+(r+s \cdot x+h \cdot x^{\mathbf{2}})^{\mathbf{2}}}$.

(115) Suppose $Z \subseteq \mathrm{dom}((\text{the function arctan}) \cdot (\text{the function exp}))$ and for every x such that $x \in Z$ holds $\exp x < 1$. Then

(i) (the function arctan) $\cdot$(the function exp) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function arctan}) \cdot (\text{the function exp}))'_{\restriction Z}(x) = \frac{\exp x}{1+(\exp x)^{\mathbf{2}}}$.

(116) Suppose $Z \subseteq \mathrm{dom}((\text{the function arccot}) \cdot (\text{the function exp}))$ and for every x such that $x \in Z$ holds $\exp x < 1$. Then

(i) (the function arccot) $\cdot$(the function exp) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function arccot}) \cdot (\text{the function exp}))'_{\restriction Z}(x) = -\frac{\exp x}{1+(\exp x)^{\mathbf{2}}}$.

(117) Suppose that

(i) $Z \subseteq \mathrm{dom}((\text{the function arctan}) \cdot (\text{the function ln}))$, and

(ii) for every x such that $x \in Z$ holds $-1 <$ (the function ln)(x) and (the function ln)$(x) < 1$.

Then

(iii) (the function arctan) $\cdot$(the function ln) is differentiable on Z, and
(iv) for every x such that $x \in Z$ holds ((the function arctan) $\cdot$(the function ln))$'_{\restriction Z}(x) = \frac{1}{x \cdot (1+(\text{the function ln})(x)^{\mathbf{2}})}$.

(118) Suppose that
(i) $Z \subseteq$ dom((the function arccot) $\cdot$(the function ln)), and
(ii) for every x such that $x \in Z$ holds $-1 <$ (the function ln)(x) and (the function ln)$(x) < 1$.
Then
(iii) (the function arccot) $\cdot$(the function ln) is differentiable on Z, and
(iv) for every x such that $x \in Z$ holds ((the function arccot) $\cdot$(the function ln))$'_{\restriction Z}(x) = -\frac{1}{x \cdot (1+(\text{the function ln})(x)^{\mathbf{2}})}$.

(119) Suppose $Z \subseteq$ dom((the function exp) $\cdot$(the function arctan)) and $Z \subseteq]-1, 1[$. Then
(i) (the function exp) $\cdot$(the function arctan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function exp) $\cdot$(the function arctan))$'_{\restriction Z}(x) = \frac{\exp \arctan x}{1+x^{\mathbf{2}}}$.

(120) Suppose $Z \subseteq$ dom((the function exp) $\cdot$(the function arccot)) and $Z \subseteq]-1, 1[$. Then
(i) (the function exp) $\cdot$(the function arccot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function exp) $\cdot$(the function arccot))$'_{\restriction Z}(x) = -\frac{\exp \operatorname{arccot} x}{1+x^{\mathbf{2}}}$.

(121) Suppose $Z \subseteq$ dom((the function arctan)$-\mathrm{id}_Z$) and $Z \subseteq]-1, 1[$. Then
(i) (the function arctan)$-\mathrm{id}_Z$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function arctan)$-\mathrm{id}_Z$)$'_{\restriction Z}(x) = -\frac{x^{\mathbf{2}}}{1+x^{\mathbf{2}}}$.

(122) Suppose $Z \subseteq$ dom($-$the function arccot $-$ id_Z) and $Z \subseteq]-1, 1[$. Then
(i) $-$the function arccot $-$ id_Z is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ($-$the function arccot $-$ id_Z)$'_{\restriction Z}(x) = -\frac{x^{\mathbf{2}}}{1+x^{\mathbf{2}}}$.

(123) Suppose $Z \subseteq]-1, 1[$. Then
(i) (the function exp) (the function arctan) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function exp) (the function arctan))$'_{\restriction Z}(x) = \exp x \cdot \arctan x + \frac{\exp x}{1+x^{\mathbf{2}}}$.

(124) Suppose $Z \subseteq]-1, 1[$. Then
(i) (the function exp) (the function arccot) is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds ((the function exp) (the function arccot))$'_{\restriction Z}(x) = \exp x \cdot \operatorname{arccot} x - \frac{\exp x}{1+x^{\mathbf{2}}}$.

(125) Suppose $Z \subseteq$ dom($\frac{1}{r}$ ((the function arctan) $\cdot f$) $-$ id_Z) and for every x such that $x \in Z$ holds $f(x) = r \cdot x$ and $r \neq 0$ and $-1 < f(x) < 1$. Then
(i) $\frac{1}{r}$ ((the function arctan) $\cdot f$) $-$ id_Z is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{r}\,((\text{the function arctan}) \cdot f) - \mathrm{id}_Z)'_{\restriction Z}(x) = -\frac{(r \cdot x)^2}{1+(r \cdot x)^2}$.

(126) Suppose $Z \subseteq \mathrm{dom}((-\frac{1}{r})\,((\text{the function arccot}) \cdot f) - \mathrm{id}_Z)$ and for every x such that $x \in Z$ holds $f(x) = r \cdot x$ and $r \neq 0$ and $-1 < f(x) < 1$. Then

(i) $(-\frac{1}{r})\,((\text{the function arccot}) \cdot f) - \mathrm{id}_Z$ is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((-\frac{1}{r})\,((\text{the function arccot}) \cdot f) - \mathrm{id}_Z)'_{\restriction Z}(x) = -\frac{(r \cdot x)^2}{1+(r \cdot x)^2}$.

(127) Suppose $Z \subseteq \mathrm{dom}((\text{the function ln})\ (\text{the function arctan}))$ and $Z \subseteq]-1, 1[$. Then

(i) (the function ln) (the function arctan) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function ln})\ (\text{the function arctan}))'_{\restriction Z}(x) = \frac{\arctan x}{x} + \frac{(\text{the function ln})(x)}{1+x^2}$.

(128) Suppose $Z \subseteq \mathrm{dom}((\text{the function ln})\ (\text{the function arccot}))$ and $Z \subseteq]-1, 1[$. Then

(i) (the function ln) (the function arccot) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $((\text{the function ln})\ (\text{the function arccot}))'_{\restriction Z}(x) = \frac{\mathrm{arccot}\, x}{x} - \frac{(\text{the function ln})(x)}{1+x^2}$.

(129) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{f}\ \text{the function arctan})$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $f(x) = x$. Then

(i) $\frac{1}{f}$ the function arctan is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\ \text{the function arctan})'_{\restriction Z}(x) = -\frac{\arctan x}{x^2} + \frac{1}{x \cdot (1+x^2)}$.

(130) Suppose $Z \subseteq \mathrm{dom}(\frac{1}{f}\ \text{the function arccot})$ and $Z \subseteq]-1, 1[$ and for every x such that $x \in Z$ holds $f(x) = x$. Then

(i) $\frac{1}{f}$ the function arccot is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\frac{1}{f}\ \text{the function arccot})'_{\restriction Z}(x) = -\frac{\mathrm{arccot}\, x}{x^2} - \frac{1}{x \cdot (1+x^2)}$.

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