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Riemann Integral of Functions $\mathbb{R}$ into $\mathbb{C}$

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Summary. In this article, we define the Riemann Integral on functions $\mathbb{R}$ into $\mathbb{C}$ and proof the linearity of this operator. Especially, the Riemann integral of complex functions is constituted by the redefinition about the Riemann sum of complex numbers. Our method refers to the [19].

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The terminology and notation used here have been introduced in the following articles: [5], [1], [16], [18], [4], [6], [7], [15], [10], [13], [11], [12], [2], [3], [8], [17], [21], [9], [14], and [20].

1. Preliminaries

One can prove the following proposition

(1) For every complex number z and for every real number r holds $\Re(r \cdot z) = r \cdot \Re(z)$ and $\Im(r \cdot z) = r \cdot \Im(z)$.

Let S be a finite sequence of elements of $\mathbb{C}$. The functor $\Re(S)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 1) $\Re(S) = \Re(S$ **qua** partial function from $\mathbb{N}$ to $\mathbb{C})$.

The functor $\Im(S)$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 2) $\Im(S) = \Im(S$ **qua** partial function from $\mathbb{N}$ to $\mathbb{C})$.

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Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, let S be a non empty Division of A, and let D be an element of S. A finite sequence of elements of $\mathbb{C}$ is said to be a middle volume of f and D if it satisfies the conditions (Def. 3).

(Def. 3)(i) $\operatorname{len} \mathrm{it} = \operatorname{len} D$, and

(ii) for every natural number i such that $i \in \operatorname{dom} D$ there exists an element c of $\mathbb{C}$ such that $c \in \operatorname{rng}(f \restriction \operatorname{divset}(D, i))$ and $\mathrm{it}(i) = c \cdot \operatorname{vol}(\operatorname{divset}(D, i))$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, let S be a non empty Division of A, let D be an element of S, and let F be a middle volume of f and D. The functor middle sum(f, F) yields an element of $\mathbb{C}$ and is defined by:

(Def. 4) middle sum$(f, F) = \sum F$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, and let T be a DivSequence of A. A function from $\mathbb{N}$ into $\mathbb{C}^*$ is said to be a middle volume sequence of f and T if:

(Def. 5) For every element k of $\mathbb{N}$ holds $\mathrm{it}(k)$ is a middle volume of f and $T(k)$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, let T be a DivSequence of A, let S be a middle volume sequence of f and T, and let k be an element of $\mathbb{N}$. Then $S(k)$ is a middle volume of f and $T(k)$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a function from A into $\mathbb{C}$, let T be a DivSequence of A, and let S be a middle volume sequence of f and T. The functor middle sum(f, S) yields a complex sequence and is defined as follows:

(Def. 6) For every element i of $\mathbb{N}$ holds (middle sum$(f, S))(i) =$ middle sum$(f, S(i))$.

2. Definition of Riemann Integral of Functions $\mathbb{R}$ into $\mathbb{C}$

Next we state two propositions:

(2) For every partial function f from $\mathbb{R}$ to $\mathbb{C}$ and for every subset A of $\mathbb{R}$ holds $\Re(f \restriction A) = \Re(f) \restriction A$.

(3) For every partial function f from $\mathbb{R}$ to $\mathbb{C}$ and for every subset A of $\mathbb{R}$ holds $\Im(f \restriction A) = \Im(f) \restriction A$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a function from A into $\mathbb{C}$. Observe that $\Re(f)$ is quasi total and $\Im(f)$ is quasi total.

We now state several propositions:

(4) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{C}$, s be a non empty Division of A, D be an element of s, and S be a middle volume of f and D. Then $\Re(S)$ is a middle volume of $\Re(f)$ and D and $\Im(S)$ is a middle volume of $\Im(f)$ and D.

(5) For every finite sequence F of elements of $\mathbb{C}$ and for every element x of $\mathbb{C}$ holds $\Re(F \frown \langle x \rangle) = \Re(F) \frown \langle \Re(x) \rangle$.

(6) For every finite sequence F of elements of $\mathbb{C}$ and for every element x of $\mathbb{C}$ holds $\Im(F \frown \langle x \rangle) = \Im(F) \frown \langle \Im(x) \rangle$.

(7) Let F be a finite sequence of elements of $\mathbb{C}$ and F_1 be a finite sequence of elements of $\mathbb{R}$. If $F_1 = \Re(F)$, then $\sum F_1 = \Re(\sum F)$.

(8) Let F be a finite sequence of elements of $\mathbb{C}$ and F_2 be a finite sequence of elements of $\mathbb{R}$. If $F_2 = \Im(F)$, then $\sum F_2 = \Im(\sum F)$.

(9) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{C}$, S be a non empty Division of A, D be an element of S, F be a middle volume of f and D, and F_1 be a middle volume of $\Re(f)$ and D. If $F_1 = \Re(F)$, then $\Re(\text{middle sum}(f, F)) = \text{middle sum}(\Re(f), F_1)$.

(10) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{C}$, S be a non empty Division of A, D be an element of S, F be a middle volume of f and D, and F_2 be a middle volume of $\Im(f)$ and D. If $F_2 = \Im(F)$, then $\Im(\text{middle sum}(f, F)) = \text{middle sum}(\Im(f), F_2)$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a function from A into $\mathbb{C}$. We say that f is integrable if and only if:

(Def. 7) $\Re(f)$ is integrable and $\Im(f)$ is integrable.

We now state three propositions:

(11) For every partial function f from $\mathbb{R}$ to $\mathbb{C}$ holds f is bounded iff $\Re(f)$ is bounded and $\Im(f)$ is bounded.

(12) Let A be a non empty subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, and g be a function from A into $\mathbb{C}$. If $f = g$, then $\Re(f) = \Re(g)$ and $\Im(f) = \Im(g)$.

(13) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{C}$. Then f is bounded if and only if $\Re(f)$ is bounded and $\Im(f)$ is bounded.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a function from A into $\mathbb{C}$. The functor integral f yielding an element of $\mathbb{C}$ is defined as follows:

(Def. 8) $\text{integral}\, f = \text{integral}\, \Re(f) + \text{integral}\, \Im(f) \cdot i$.

Next we state two propositions:

(14) Let A be a closed-interval subset of $\mathbb{R}$, f be a function from A into $\mathbb{C}$, T be a DivSequence of A, and S be a middle volume sequence of f and T. Suppose f is bounded and integrable and δ_T is convergent and $\lim(\delta_T) = 0$. Then middle sum(f, S) is convergent and $\lim \text{middle sum}(f, S) = \text{integral}\, f$.

(15) Let A be a closed-interval subset of $\mathbb{R}$ and f be a function from A into $\mathbb{C}$. Suppose f is bounded. Then f is integrable if and only if there exists an element I of $\mathbb{C}$ such that for every DivSequence T of A and for every middle volume sequence S of f and T such that δ_T is convergent and $\lim(\delta_T) = 0$

holds middle sum(f, S) is convergent and lim middle sum$(f, S) = I$.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. We say that f is integrable on A if and only if:

(Def. 9) $\Re(f)$ is integrable on A and $\Im(f)$ is integrable on A.

Let A be a closed-interval subset of $\mathbb{R}$ and let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. The functor $\int_A f(x)dx$ yields an element of $\mathbb{C}$ and is defined by:

(Def. 10) $\int_A f(x)dx = \int_A \Re(f)(x)dx + \int_A \Im(f)(x)dx \cdot i$.

We now state two propositions:

(16) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, and g be a function from A into $\mathbb{C}$. Suppose $f \restriction A = g$. Then f is integrable on A if and only if g is integrable.

(17) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, and g be a function from A into $\mathbb{C}$. If $f \restriction A = g$, then $\int_A f(x)dx =$ integral g.

Let a, b be real numbers and let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. The functor $\int_a^b f(x)dx$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 11) $\int_a^b f(x)dx = \int_a^b \Re(f)(x)dx + \int_a^b \Im(f)(x)dx \cdot i$.

3. Linearity of the Integration Operator

Next we state several propositions:

(18) Let c be a complex number and f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. Then $\Re(c\,f) = \Re(c)\,\Re(f) - \Im(c)\,\Im(f)$ and $\Im(c\,f) = \Re(c)\,\Im(f) + \Im(c)\,\Re(f)$.

(19) Let A be a closed-interval subset of $\mathbb{R}$ and f_1, f_2 be partial functions from $\mathbb{R}$ to $\mathbb{C}$. Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \operatorname{dom} f_1$ and $A \subseteq \operatorname{dom} f_2$ and $f_1 \restriction A$ is bounded and $f_2 \restriction A$ is bounded. Then $f_1 + f_2$ is integrable on A and $f_1 - f_2$ is integrable on A and $\int_A (f_1 + f_2)(x)dx = \int_A f_1(x)dx + \int_A f_2(x)dx$ and $\int_A (f_1 - f_2)(x)dx = \int_A f_1(x)dx - \int_A f_2(x)dx$.

(20) Let r be a real number, A be a closed-interval subset of $\mathbb{R}$, and f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. Suppose $A \subseteq \operatorname{dom} f$ and f is integrable on

A and $f\restriction A$ is bounded. Then $r\,f$ is integrable on A and $\int\limits_{A}(r\,f)(x)dx = r \cdot \int\limits_{A} f(x)dx$.

(21) Let c be a complex number, A be a closed-interval subset of $\mathbb{R}$, and f be a partial function from $\mathbb{R}$ to $\mathbb{C}$. Suppose $A \subseteq \operatorname{dom} f$ and f is integrable on A and $f\restriction A$ is bounded. Then $c\,f$ is integrable on A and $\int\limits_{A}(c\,f)(x)dx = c \cdot \int\limits_{A} f(x)dx$.

(22) Let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [a, b]$, then $\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

(23) Let f be a partial function from $\mathbb{R}$ to $\mathbb{C}$, A be a closed-interval subset of $\mathbb{R}$, and a, b be real numbers. If $A = [b, a]$, then $-\int\limits_{A} f(x)dx = \int\limits_{a}^{b} f(x)dx$.

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