Brouwer Fixed Point Theorem in the General Case

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Summary. In this article we prove the Brouwer fixed point theorem for an arbitrary convex compact subset of $\mathbb{E}^n$ with a non empty interior. This article is based on [15].

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The notation and terminology used here have been introduced in the following papers: [17], [12], [1], [4], [7], [16], [6], [13], [10], [2], [3], [14], [9], [20], [18], [8], [19], [11], [21], and [5].

1. Preliminaries

For simplicity, we adopt the following convention: $n$ is a natural number, $p$, $q$, $u$, $w$ are points of $\mathbb{E}^n_T$, $S$ is a subset of $\mathbb{E}^n_T$, $A$, $B$ are convex subsets of $\mathbb{E}^n_T$, and $r$ is a real number.

Next we state several propositions:

1. $(1-r) \cdot p + r \cdot q = p + r \cdot (q - p)$.
2. If $u, w \in \text{halfline}(p, q)$ and $|u - p| = |w - p|$, then $u = w$.
3. Let given $S$. Suppose $p \in S$ and $p \neq q$ and $S \cap \text{halfline}(p, q)$ is Bounded. Then there exists $w$ such that
   (i) $w \in \text{Fr} S \cap \text{halfline}(p, q)$,
   (ii) for every $u$ such that $u \in S \cap \text{halfline}(p, q)$ holds $|p - u| \leq |p - w|$, and
   (iii) for every $r$ such that $r > 0$ there exists $u$ such that $u \in S \cap \text{halfline}(p, q)$ and $|w - u| < r$. 

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(4) For every $A$ such that $A$ is closed and $p \in \text{Int} A$ and $p \neq q$ and $A \cap \text{halfline}(p, q)$ is Bounded there exists $u$ such that $\text{Fr} A \cap \text{halfline}(p, q) = \{u\}$.

(5) If $r > 0$, then $\text{Fr Ball}(p, r) = \text{Sphere}(p, r)$.

Let $n$ be an element of $\mathbb{N}$, let $A$ be a Bounded subset of $E^n_T$, and let $p$ be a point of $E^n_T$. One can verify that $p + A$ is Bounded.

2. Main Theorems

Next we state four propositions:

(6) Let $n$ be an element of $\mathbb{N}$ and $A$ be a convex subset of $E^n_T$. Suppose $A$ is compact and non boundary. Then there exists a function $h$ from $E^n_T \setminus A$ into $\text{Tdisk}(0_{E^n_T}, 1)$ such that $h$ is homeomorphism and $h \circ \text{Fr} A = \text{Sphere}(0_{E^n_T}, 1)$.

(7) Let given $A$, $B$. Suppose $A$ is compact and non boundary and $B$ is compact and non boundary. Then there exists a function $h$ from $E^n_T \setminus A$ into $E^n_T \setminus B$ such that $h$ is homeomorphism and $h \circ \text{Fr} A = \text{Fr} B$.

(8)$^1$ For every $A$ such that $A$ is compact and non boundary holds every continuous function from $E^n_T \setminus A$ into $E^n_T \setminus A$ has a fixpoint.

(9) Let $A$ be a non empty convex subset of $E^n_T$. Suppose $A$ is compact and non boundary. Let $F_1$ be a non empty subspace of $E^n_T \setminus A$. If $\Omega(F_1) = \text{Fr} A$, then $F_1$ is not a retract of $E^n_T \setminus A$.

References


$^1$Brouwer Fixed Point Theorem

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