

# First Order Languages: Further Syntax and Semantics<sup>1</sup>

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**Summary.** Third of a series of articles laying down the bases for classical first order model theory. Interpretation of a language in a universe set. Evaluation of a term in a universe. Truth evaluation of an atomic formula. Reassigning the value of a symbol in a given interpretation. Syntax and semantics of a non atomic formula are then defined concurrently (this point is explained in [16], 4.2.1). As a consequence, the evaluation of any w.f.f. string and the relation of logical implication are introduced. Depth of a formula. Definition of satisfaction and entailment (aka entailment or logical implication) relations, see [18] III.3.2 and III.4.1 respectively.

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The terminology and notation used in this paper have been introduced in the following papers: [7], [1], [23], [6], [8], [17], [14], [15], [22], [9], [10], [11], [2], [21], [26], [24], [5], [3], [4], [12], [27], [28], [19], [20], [25], and [13].

For simplicity, we follow the rules:  $m, n$  denote natural numbers,  $m_1$  denotes an element of  $\mathbb{N}$ ,  $A, B, X, Y, Z, x, y$  denote sets,  $S, S_1, S_2$  denote languages,  $s$  denotes an element of  $S$ ,  $w, w_1, w_2$  denote strings of  $S$ ,  $U$  denotes a non empty set,  $f, g$  denote functions, and  $p, p_2$  denote finite sequences.

Let us consider  $S$ . Then  $\text{TheNorSymbOf } S$  is an element of  $S$ .

Let  $U$  be a non empty set. The functor  $U\text{-deltaInterpreter}$  yielding a function from  $U^2$  into *Boolean* is defined by:

(Def. 1)  $U\text{-deltaInterpreter} = \chi_{(\text{the concatenation of } U)^\circ(\text{id}_{U^1}), U^2}$ .

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Let  $X$  be a set. Then  $\text{id}_X$  is an equivalence relation of  $X$ .

Let  $S$  be a language, let  $U$  be a non empty set, and let  $s$  be an of-atomic-formula element of  $S$ . Interpreter of  $s$  and  $U$  is defined as follows:

- (Def. 2)(i) It is a function from  $U^{|\text{ar } s|}$  into *Boolean* if  $s$  is relational,  
(ii) it is a function from  $U^{|\text{ar } s|}$  into  $U$ , otherwise.

Let us consider  $S, U$  and let  $s$  be an of-atomic-formula element of  $S$ . We see that the interpreter of  $s$  and  $U$  is a function from  $U^{|\text{ar } s|}$  into  $U \cup \textit{Boolean}$ .

Let us consider  $S, U$  and let  $s$  be a termal element of  $S$ . One can verify that every interpreter of  $s$  and  $U$  is  $U$ -valued.

Let  $S$  be a language. Note that every element of  $S$  which is literal is also own.

Let us consider  $S, U$ . A function is called an interpreter of  $S$  and  $U$  if:

- (Def. 3) For every own element  $s$  of  $S$  holds  $\text{it}(s)$  is an interpreter of  $s$  and  $U$ .

Let us consider  $S, U, f$ . We say that  $f$  is  $(S, U)$ -interpreter-like if and only if:

- (Def. 4)  $f$  is an interpreter of  $S$  and  $U$  and function yielding.

Let us consider  $S$  and let  $U$  be a non empty set. One can verify that every function which is  $(S, U)$ -interpreter-like is also function yielding.

Let us consider  $S, U$  and let  $s$  be an own element of  $S$ . Observe that every interpreter of  $s$  and  $U$  is non empty.

Let  $S$  be a language and let  $U$  be a non empty set. Note that there exists a function which is  $(S, U)$ -interpreter-like.

Let us consider  $S, U$ , let  $I$  be an  $(S, U)$ -interpreter-like function, and let  $s$  be an own element of  $S$ . Then  $I(s)$  is an interpreter of  $s$  and  $U$ .

Let  $S$  be a language, let  $U$  be a non empty set, let  $I$  be an  $(S, U)$ -interpreter-like function, let  $x$  be an own element of  $S$ , and let  $f$  be an interpreter of  $x$  and  $U$ . One can check that  $I+\cdot(x\mapsto f)$  is  $(S, U)$ -interpreter-like.

Let us consider  $f, x, y$ . The functor  $(x, y)$  ReassignIn  $f$  yields a function and is defined by:

- (Def. 5)  $(x, y)$  ReassignIn  $f = f+\cdot(x\mapsto(\emptyset\mapsto y))$ .

Let  $S$  be a language, let  $U$  be a non empty set, let  $I$  be an  $(S, U)$ -interpreter-like function, let  $x$  be a literal element of  $S$ , and let  $u$  be an element of  $U$ . One can verify that  $(x, u)$  ReassignIn  $I$  is  $(S, U)$ -interpreter-like.

Let  $S$  be a language. One can check that AllSymbolsOf  $S$  is non empty.

Let  $Y$  be a set and let  $X, Z$  be non empty sets. Observe that every function from  $X$  into  $Z^Y$  is function yielding.

Let  $X, Y, Z$  be non empty sets. One can verify that there exists a function from  $X$  into  $Z^Y$  which is function yielding.

Let  $f$  be a function yielding function and let  $g$  be a function. The functor  $[g, f]$  yields a function and is defined by:

(Def. 6)  $\text{dom}[g, f] = \text{dom } f$  and for every  $x$  such that  $x \in \text{dom } f$  holds  $[g, f](x) = g \cdot f(x)$ .

Let  $f$  be an empty function and let  $g$  be a function. One can verify that  $[g, f]$  is empty.

Let  $f$  be a function yielding function and let  $g$  be a function. The functor  $[f, g]$  yielding a function is defined as follows:

(Def. 7)  $\text{dom}[f, g] = \text{dom } f \cap \text{dom } g$  and for every set  $x$  such that  $x \in \text{dom}[f, g]$  holds  $[f, g](x) = f(x)(g(x))$ .

Let  $f$  be a function yielding function and let  $g$  be an empty function. One can check that  $[f, g]$  is empty.

Let  $X$  be a finite sequence-membered set. Observe that every function which is  $X$ -valued is also function yielding.

Let  $E, D$  be non empty sets, let  $p$  be a  $D$ -valued finite sequence, and let  $h$  be a function from  $D$  into  $E$ . Note that  $h \cdot p$  is  $\text{len } p$ -element.

Let  $X, Y$  be non empty sets, let  $f$  be a function from  $X$  into  $Y$ , and let  $p$  be an  $X$ -valued finite sequence. One can verify that  $f \cdot p$  is finite sequence-like.

Let  $E, D$  be non empty sets, let  $n$  be a natural number, let  $p$  be an  $n$ -element  $D$ -valued finite sequence, and let  $h$  be a function from  $D$  into  $E$ . Observe that  $h \cdot p$  is  $n$ -element.

We now state the proposition

(1) For every 0-terminal string  $t_0$  of  $S$  holds  $t_0 = \langle S\text{-firstChar}(t_0) \rangle$ .

Let us consider  $S$ , let  $U$  be a non empty set, let  $u$  be an element of  $U$ , and let  $I$  be an  $(S, U)$ -interpreter-like function. The functor  $(I, u)\text{-TermEval}$  yields a function from  $\mathbb{N}$  into  $U^{\text{AllTermsOf } S}$  and is defined as follows:

(Def. 8)  $(I, u)\text{-TermEval}(0) = \text{AllTermsOf } S \mapsto u$  and for every  $m_1$  holds  $(I, u)\text{-TermEval}(m_1 + 1) = [I \cdot S\text{-firstChar}, [(I, u)\text{-TermEval}(m_1) \text{ qua function}, S\text{-subTerms}]]$ .

Let us consider  $S, U$ , let  $I$  be an  $(S, U)$ -interpreter-like function, and let  $t$  be an element of  $\text{AllTermsOf } S$ . The functor  $I\text{-TermEval } t$  yields an element of  $U$  and is defined as follows:

(Def. 9) For every element  $u_1$  of  $U$  and for every  $m_1$  such that  $t \in S\text{-termsOfMaxDepth}(m_1)$  holds  $I\text{-TermEval } t = (I, u_1)\text{-TermEval}(m_1 + 1)(t)$ .

Let us consider  $S, U$  and let  $I$  be an  $(S, U)$ -interpreter-like function. The functor  $I\text{-TermEval}$  yielding a function from  $\text{AllTermsOf } S$  into  $U$  is defined by:

(Def. 10) For every element  $t$  of  $\text{AllTermsOf } S$  holds  $I\text{-TermEval}(t) = I\text{-TermEval } t$ .

Let us consider  $S, U$  and let  $I$  be an  $(S, U)$ -interpreter-like function. The functor  $I\text{====}$  yielding a function is defined as follows:

(Def. 11)  $I\text{====} = I + \cdot (\text{TheEqSymbOf } S \mapsto U\text{-deltaInterpreter})$ .

Let us consider  $S, U$ , let  $I$  be an  $(S, U)$ -interpreter-like function, and let  $x$  be a set. We say that  $x$  is  $I$ -extension if and only if:

(Def. 12)  $x = I == = .$

Let us consider  $S, U$  and let  $I$  be an  $(S, U)$ -interpreter-like function. Note that  $I == =$  is  $I$ -extension and every set which is  $I$ -extension is also function-like. Observe that there exists a function which is  $I$ -extension. Observe that  $I == =$  is  $(S, U)$ -interpreter-like.

Let  $f$  be an  $I$ -extension function, and let  $s$  be an of-atomic-formula element of  $S$ . Then  $f(s)$  is an interpreter of  $s$  and  $U$ .

Let  $p_1$  be a 0-w.f.f. string of  $S$ . The functor  $I$ -AtomicEval  $p_1$  is defined as follows:

(Def. 13)  $I$ -AtomicEval  $p_1 = (I == = (S\text{-firstChar}(p_1)))(I\text{-TermEval} \cdot \text{SubTerms } p_1)$ .

Let us consider  $S, U$ , let  $I$  be an  $(S, U)$ -interpreter-like function, and let  $p_1$  be a 0-w.f.f. string of  $S$ . Then  $I$ -AtomicEval  $p_1$  is an element of  $Boolean$ . Note that  $I \upharpoonright \text{OwnSymbolsOf } S$  is  $(U^* \rightarrow (U \cup Boolean))$ -valued and  $I \upharpoonright \text{OwnSymbolsOf } S$  is  $(S, U)$ -interpreter-like.

Let us consider  $S, U$  and let  $I$  be an  $(S, U)$ -interpreter-like function. Observe that  $I \upharpoonright \text{OwnSymbolsOf } S$  is total.

Let us consider  $S, U$ . The functor  $U$ -InterpretersOf  $S$  is defined by:

(Def. 14)  $U$ -InterpretersOf  $S = \{f \in (U^* \rightarrow (U \cup Boolean))^{\text{OwnSymbolsOf } S} : f \text{ is } (S, U)\text{-interpreter-like}\}$ .

Let us consider  $S, U$ . Then  $U$ -InterpretersOf  $S$  is a subset of  $(U^* \rightarrow (U \cup Boolean))^{\text{OwnSymbolsOf } S}$ . Observe that  $U$ -InterpretersOf  $S$  is non empty. One can verify that every element of  $U$ -InterpretersOf  $S$  is  $(S, U)$ -interpreter-like. The functor  $S$ -TruthEval  $U$  yields a function from

$(U\text{-InterpretersOf } S) \times \text{AtomicFormulasOf } S$  into  $Boolean$  and is defined by:

(Def. 15) For every element  $I$  of  $U$ -InterpretersOf  $S$  and for every element  $p_1$  of  $\text{AtomicFormulasOf } S$  holds  $(S\text{-TruthEval } U)(I, p_1) = I\text{-AtomicEval } p_1$ .

Let us consider  $S, U$ , let  $I$  be an element of  $U$ -InterpretersOf  $S$ , let  $f$  be a partial function from  $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$  to  $Boolean$ , and let  $p_1$  be an element of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ . The functor  $f$ -ExFunctor  $(I, p_1)$  yielding an element of  $Boolean$  is defined as follows:

(Def. 16)  $f$ -ExFunctor  $(I, p_1) = \begin{cases} true, & \text{if there exists an element } u \text{ of } U \text{ and} \\ & \text{there exists a literal element } v \text{ of } S \text{ such} \\ & \text{that } p_1(1) = v \text{ and} \\ & f((v, u) \text{ ReassignIn } I, (p_1) \downarrow_1) = true, \\ false, & \text{otherwise.} \end{cases}$

Let us consider  $S, U$  and let  $g$  be an element of  $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \rightarrow Boolean$ . The functor ExIterator  $g$  yields a partial function from  $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$  to  $Boolean$  and

is defined by the conditions (Def. 17).

- (Def. 17)(i) For every element  $x$  of  $U$ -InterpretersOf  $S$  and for every element  $y$  of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$  holds  $\langle x, y \rangle \in \text{dom ExIterator } g$  iff there exists a literal element  $v$  of  $S$  and there exists a string  $w$  of  $S$  such that  $\langle x, w \rangle \in \text{dom } g$  and  $y = \langle v \rangle \wedge w$ , and
- (ii) for every element  $x$  of  $U$ -InterpretersOf  $S$  and for every element  $y$  of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$  such that  $\langle x, y \rangle \in \text{dom ExIterator } g$  holds  $(\text{ExIterator } g)(x, y) = g\text{-ExFunctor}(x, y)$ .

Let us consider  $S, U$ , let  $f$  be a partial function from  $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$  to  $Boolean$ , let  $I$  be an element of  $U\text{-InterpretersOf } S$ , and let  $p_1$  be an element of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ .

The functor  $f\text{-NorFunctor}(I, p_1)$  yielding an element of  $Boolean$  is defined by:

$$(Def. 18) \quad f\text{-NorFunctor}(I, p_1) = \begin{cases} true, & \text{if there exist elements } w_1, w_2 \text{ of} \\ & (\text{AllSymbolsOf } S)^* \setminus \{\emptyset\} \text{ such that} \\ & \langle I, w_1 \rangle \in \text{dom } f \text{ and } f(I, w_1) = false \\ & \text{and } f(I, w_2) = false \text{ and} \\ & p_1 = \langle \text{TheNorSymbOf } S \rangle \wedge w_1 \wedge w_2, \\ false, & \text{otherwise.} \end{cases}$$

Let us consider  $S, U$  and let  $g$  be an element of  $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \rightarrow Boolean$ . The functor  $\text{NorIterator } g$  yielding a partial function from  $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\})$  to  $Boolean$  is defined by the conditions (Def. 19).

- (Def. 19)(i) For every element  $x$  of  $U$ -InterpretersOf  $S$  and for every element  $y$  of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$  holds  $\langle x, y \rangle \in \text{dom NorIterator } g$  iff there exist elements  $p_3, p_4$  of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$  such that  $y = \langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$  and  $\langle x, p_3 \rangle, \langle x, p_4 \rangle \in \text{dom } g$ , and
- (ii) for every element  $x$  of  $U$ -InterpretersOf  $S$  and for every element  $y$  of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$  such that  $\langle x, y \rangle \in \text{dom NorIterator } g$  holds  $(\text{NorIterator } g)(x, y) = g\text{-NorFunctor}(x, y)$ .

Let us consider  $S, U$ . The functor  $(S, U)\text{-TruthEval}$  yields a function from  $\mathbb{N}$  into  $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \rightarrow Boolean$  and is defined as follows:

- (Def. 20)  $(S, U)\text{-TruthEval}(0) = S\text{-TruthEval } U$  and for every  $m_1$  holds  $(S, U)\text{-TruthEval}(m_1+1) = \text{ExIterator}(S, U)\text{-TruthEval}(m_1) + \cdot \text{NorIterator}(S, U)\text{-TruthEval}(m_1) + \cdot (S, U)\text{-TruthEval}(m_1)$ .

Next we state the proposition

- (2) For every  $(S, U)$ -interpreter-like function  $I$  holds  $I \upharpoonright \text{OwnSymbolsOf } S \in U\text{-InterpretersOf } S$ .

Let  $S$  be a language, let  $m$  be a natural number, and let  $U$  be a non empty set.

The functor  $(S, U)$ -TruthEval  $m$  yielding an element of  $(U\text{-InterpretersOf } S) \times ((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \rightarrow \text{Boolean}$  is defined as follows:

(Def. 21) For every  $m_1$  such that  $m = m_1$  holds  $(S, U)$ -TruthEval  $m = (S, U)$ -TruthEval( $m_1$ ).

Let us consider  $S, U, m$  and let  $I$  be an element of  $U\text{-InterpretersOf } S$ . The functor  $(I, m)$ -TruthEval yields an element of

$((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \rightarrow \text{Boolean}$  and is defined by:

(Def. 22)  $(I, m)$ -TruthEval =  $(\text{curry}((S, U)\text{-TruthEval } m))(I)$ .

Let us consider  $S, m$ . The functor  $S$ -formulasOfMaxDepth  $m$  yielding a subset of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$  is defined as follows:

(Def. 23) For every non empty set  $U$  and for every element  $I$  of  $U\text{-InterpretersOf } S$  and for every element  $m_1$  of  $\mathbb{N}$  such that  $m = m_1$  holds  $S$ -formulasOfMaxDepth  $m = \text{dom}((I, m_1)\text{-TruthEval})$ .

Let us consider  $S, m, w$ . We say that  $w$  is  $m$ -w.f.f. if and only if:

(Def. 24)  $w \in S$ -formulasOfMaxDepth  $m$ .

Let us consider  $S, w$ . We say that  $w$  is w.f.f. if and only if:

(Def. 25) There exists  $m$  such that  $w$  is  $m$ -w.f.f..

Let us consider  $S$ . Note that every string of  $S$  which is 0-w.f.f. is also 0-w.f.f. and every string of  $S$  which is 0-w.f.f. is also 0-w.f.f.. Let us consider  $m$ . One can check that every string of  $S$  which is  $m$ -w.f.f. is also w.f.f.. Let us consider  $n$ . One can check that every string of  $S$  which is  $m + 0 \cdot n$ -w.f.f. is also  $m + n$ -w.f.f..

Let us consider  $S, m$ . Observe that there exists a string of  $S$  which is  $m$ -w.f.f.. Note that  $S$ -formulasOfMaxDepth  $m$  is non empty. One can verify that there exists a string of  $S$  which is w.f.f..

Let us consider  $S, U$ , let  $I$  be an element of  $U\text{-InterpretersOf } S$ , and let  $w$  be a w.f.f. string of  $S$ . The functor  $I$ -TruthEval  $w$  yields an element of  $\text{Boolean}$  and is defined as follows:

(Def. 26) For every natural number  $m$  such that  $w$  is  $m$ -w.f.f. holds  $I$ -TruthEval  $w = (I, m)$ -TruthEval( $w$ ).

Let us consider  $S$ . The functor AllFormulasOf  $S$  is defined by:

(Def. 27) AllFormulasOf  $S = \{w; w \text{ ranges over strings of } S: \bigvee_m w \text{ is } m\text{-w.f.f.}\}$ .

Let us consider  $S$ . One can check that AllFormulasOf  $S$  is non empty.

For simplicity, we follow the rules:  $u, u_1, u_2$  are elements of  $U$ ,  $t$  is a termal string of  $S$ ,  $I$  is an  $(S, U)$ -interpreter-like function,  $l, l_1, l_2$  are literal elements of  $S$ ,  $m_2, n_1$  are non zero natural numbers,  $p_0$  is a 0-w.f.f. string of  $S$ , and  $p_5, p_1, p_3, p_4$  are w.f.f. strings of  $S$ .

The following propositions are true:

(3)  $(I, u)$ -TermEval( $m + 1$ )( $t$ ) =  $I(S\text{-firstChar}(t))((I, u)\text{-TermEval}(m) \cdot \text{SubTerms } t)$  and if  $t$  is 0-termal, then  $(I, u)\text{-TermEval}(m + 1)(t) = I(S\text{-firstChar}(t))(\emptyset)$ .

- (4) For every  $m$ -terminal string  $t$  of  $S$  holds  $(I, u_1)$ -TermEval( $m + 1$ )( $t$ ) =  $(I, u_2)$ -TermEval( $m + 1 + n$ )( $t$ ).
- (5)  $\text{curry}((S, U)\text{-TruthEval } m)$  is a function from  $U$ -InterpretersOf  $S$  into  $((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \rightarrow \text{Boolean}$ .
- (6)  $x \in X \cup Y \cup Z$  iff  $x \in X$  or  $x \in Y$  or  $x \in Z$ .
- (7)  $S$ -formulasOfMaxDepth 0 = AtomicFormulasOf  $S$ .

Let us consider  $S, m$ . Then  $S$ -formulasOfMaxDepth  $m$  can be characterized by the condition:

- (Def. 28) For every non empty set  $U$  and for every element  $I$  of  $U$ -InterpretersOf  $S$  holds  $S$ -formulasOfMaxDepth  $m = \text{dom}((I, m)\text{-TruthEval})$ .

Next we state the proposition

- (8)  $(S, U)\text{-TruthEval } m \in \text{Boolean}^{(U\text{-InterpretersOf } S) \times (S\text{-formulasOfMaxDepth } m)}$   
and  
 $(S, U)\text{-TruthEval}(m) \in \text{Boolean}^{(U\text{-InterpretersOf } S) \times (S\text{-formulasOfMaxDepth } m)}$ .

Let us consider  $S, m$ . The functor  $m$ -ExFormulasOf  $S$  is defined by:

- (Def. 29)  $m$ -ExFormulasOf  $S = \{\langle v \rangle \wedge p_1 : v \text{ ranges over elements of LettersOf } S, p_1 \text{ ranges over elements of } S\text{-formulasOfMaxDepth } m\}$ .

The functor  $m$ -NorFormulasOf  $S$  is defined as follows:

- (Def. 30)  $m$ -NorFormulasOf  $S = \{\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 : p_3 \text{ ranges over elements of } S\text{-formulasOfMaxDepth } m, p_4 \text{ ranges over elements of } S\text{-formulasOfMaxDepth } m\}$ .

Let us consider  $S$  and let  $w_1, w_2$  be strings of  $S$ . Then  $w_1 \wedge w_2$  is a string of  $S$ .

Let us consider  $S, s$ . Then  $\langle s \rangle$  is a string of  $S$ .

One can prove the following two propositions:

- (9)  $S$ -formulasOfMaxDepth( $m + 1$ ) =  $(m\text{-ExFormulasOf } S) \cup (m\text{-NorFormulasOf } S) \cup (S\text{-formulasOfMaxDepth } m)$ .
- (10) AtomicFormulasOf  $S$  is  $S$ -prefix.

Let us consider  $S$ . Note that AtomicFormulasOf  $S$  is  $S$ -prefix. Observe that  $S$ -formulasOfMaxDepth 0 is  $S$ -prefix.

Let us consider  $p_1$ . The functor Depth  $p_1$  yielding a natural number is defined by:

- (Def. 31)  $p_1$  is Depth  $p_1$ -w.f.f. and for every  $n$  such that  $p_1$  is  $n$ -w.f.f. holds Depth  $p_1 \leq n$ .

Let us consider  $S, m$  and let  $p_3, p_4$  be  $m$ -w.f.f. strings of  $S$ . Note that  $\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$  is  $m + 1$ -w.f.f..

Let us consider  $S, p_3, p_4$ . Observe that  $\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$  is w.f.f..

Let us consider  $S, m$ , let  $p_1$  be an  $m$ -w.f.f. string of  $S$ , and let  $v$  be a literal element of  $S$ . Note that  $\langle v \rangle \wedge p_1$  is  $m + 1$ -w.f.f..

Let us consider  $S, l, p_1$ . Note that  $\langle l \rangle \wedge p_1$  is w.f.f..

Let us consider  $S, w$  and let  $s$  be a non relational element of  $S$ . One can check that  $\langle s \rangle \wedge w$  is non 0-w.f.f..

Let us consider  $S, w_1, w_2$  and let  $s$  be a non relational element of  $S$ . Observe that  $\langle s \rangle \wedge w_1 \wedge w_2$  is non 0-w.f.f..

Let us consider  $S$ . Observe that  $\text{TheNorSymbOf } S$  is non relational.

Let us consider  $S, w$ . Observe that  $\langle \text{TheNorSymbOf } S \rangle \wedge w$  is non 0-w.f.f..

Let us consider  $S, l, w$ . Note that  $\langle l \rangle \wedge w$  is non 0-w.f.f..

Let us consider  $S, w$ . We say that  $w$  is exal if and only if:

(Def. 32)  $S\text{-firstChar}(w)$  is literal.

Let us consider  $S, w, l$ . One can verify that  $\langle l \rangle \wedge w$  is exal.

Let us consider  $S, m_2$ . Observe that there exists an  $m_2$ -w.f.f. string of  $S$  which is exal.

Let us consider  $S$ . Note that every string of  $S$  which is exal is also non 0-w.f.f..

Let us consider  $S, m_2$ . One can check that there exists an exal  $m_2$ -w.f.f. string of  $S$  which is non 0-w.f.f..

Let us consider  $S$ . One can verify that there exists an exal w.f.f. string of  $S$  which is non 0-w.f.f..

Let us consider  $S$  and let  $p_1$  be a non 0-w.f.f. w.f.f. string of  $S$ . Note that  $\text{Depth } p_1$  is non zero.

Let us consider  $S$  and let  $w$  be a non 0-w.f.f. w.f.f. string of  $S$ . Observe that  $S\text{-firstChar}(w)$  is non relational.

Let us consider  $S, m$ . Observe that  $S\text{-formulasOfMaxDepth } m$  is  $S$ -prefix. Then  $\text{AllFormulasOf } S$  is a subset of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ . Observe that every element of  $\text{AllFormulasOf } S$  is w.f.f.. Note that  $\text{AllFormulasOf } S$  is  $S$ -prefix.

We now state three propositions:

- (11)  $\text{dom NorIterator}((S, U)\text{-TruthEval } m) = (U\text{-InterpretersOf } S) \times (m\text{-NorFormulasOf } S)$ .
- (12)  $\text{dom ExIterator}((S, U)\text{-TruthEval } m) = (U\text{-InterpretersOf } S) \times (m\text{-ExFormulasOf } S)$ .
- (13)  $U\text{-deltaInterpreter}^{-1}(\{1\}) = \{\langle u, u \rangle : u \text{ ranges over elements of } U\}$ .

Let us consider  $S$ . Then  $\text{TheEqSymbOf } S$  is an element of  $S$ .

Let us consider  $S$ . One can verify that  $\text{ar TheEqSymbOf } S + 2$  is zero and  $|\text{ar TheEqSymbOf } S| - 2$  is zero.

We now state two propositions:

- (14) Let  $p_1$  be a 0-w.f.f. string of  $S$  and  $I$  be an  $(S, U)$ -interpreter-like function. Then
  - (i) if  $S\text{-firstChar}(p_1) \neq \text{TheEqSymbOf } S$ , then  $I\text{-AtomicEval } p_1 = I(S\text{-firstChar}(p_1))(I\text{-TermEval} \cdot \text{SubTerms } p_1)$ , and



(ii) if  $S\text{-firstChar}(p_1) = \text{TheEqSymbOf } S$ , then  $I\text{-AtomicEval } p_1 = U\text{-deltaInterpreter}(I\text{-TermEval} \cdot \text{SubTerms } p_1)$ .

(15) Let  $I$  be an  $(S, U)$ -interpreter-like function and  $p_1$  be a 0-w.f.f. string of  $S$ . If  $S\text{-firstChar}(p_1) = \text{TheEqSymbOf } S$ , then  $I\text{-AtomicEval } p_1 = 1$  iff  $I\text{-TermEval}((\text{SubTerms } p_1)(1)) = I\text{-TermEval}((\text{SubTerms } p_1)(2))$ .

Let us consider  $S, m$ . One can check that  $m\text{-ExFormulasOf } S$  is non empty. Note that  $m\text{-NorFormulasOf } S$  is non empty. Then  $m\text{-NorFormulasOf } S$  is a subset of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ .

Let us consider  $S$  and let  $w$  be an exal string of  $S$ . One can verify that  $S\text{-firstChar}(w)$  is literal.

Let us consider  $S, m$ . Observe that every element of  $m\text{-NorFormulasOf } S$  is non exal. Then  $m\text{-ExFormulasOf } S$  is a subset of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$ .

Let us consider  $S, m$ . One can check that every element of  $m\text{-ExFormulasOf } S$  is exal.

Let us consider  $S$ . One can check that there exists an element of  $S$  which is non literal.

Let us consider  $S, w$  and let  $s$  be a non literal element of  $S$ . Note that  $\langle s \rangle \wedge w$  is non exal.

Let us consider  $S, w_1, w_2$  and let  $s$  be a non literal element of  $S$ . Observe that  $\langle s \rangle \wedge w_1 \wedge w_2$  is non exal.

Let us consider  $S$ . Note that  $\text{TheNorSymbOf } S$  is non literal.

Next we state the proposition

(16)  $p_1 \in \text{AllFormulasOf } S$ .

Let us consider  $S, m, w$ . We introduce  $w$  is  $m\text{-non-w.f.f.}$  as an antonym of  $w$  is  $m\text{-w.f.f.}$ .

Let us consider  $m, S$ . One can verify that every string of  $S$  which is non  $m\text{-w.f.f.}$  is also  $m\text{-non-w.f.f.}$ .

Let us consider  $S, p_3, p_4$ . Observe that  $\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4$  is  $\max(\text{Depth } p_3, \text{Depth } p_4)\text{-non-w.f.f.}$ .

Let us consider  $S, p_1, l$ . Note that  $\langle l \rangle \wedge p_1$  is  $\text{Depth } p_1\text{-non-w.f.f.}$ .

Let us consider  $S, p_1, l$ . One can check that  $\langle l \rangle \wedge p_1$  is  $1 + \text{Depth } p_1\text{-w.f.f.}$ .

Let us consider  $S, U$ . Observe that every element of  $U\text{-InterpretersOf } S$  is  $\text{OwnSymbolsOf } S\text{-defined}$ .

Let us consider  $S, U$ . Note that there exists an element of  $U\text{-InterpretersOf } S$  which is  $\text{OwnSymbolsOf } S\text{-defined}$ .

Let us consider  $S, U$ . Note that every  $\text{OwnSymbolsOf } S\text{-defined}$  element of  $U\text{-InterpretersOf } S$  is total.

Let us consider  $S, U$ , let  $I$  be an element of  $U\text{-InterpretersOf } S$ , let  $x$  be a literal element of  $S$ , and let  $u$  be an element of  $U$ . Then  $(x, u)$   $\text{ReassignIn } I$  is an element of  $U\text{-InterpretersOf } S$ .

In the sequel  $I$  denotes an element of  $U\text{-InterpretersOf } S$ .

Let us consider  $S, w$ . The functor  $\text{xnot } w$  yields a string of  $S$  and is defined as follows:

(Def. 33)  $\text{xnot } w = \langle \text{TheNorSymbOf } S \rangle \wedge w \wedge w$ .

Let us consider  $S, m$  and let  $p_1$  be an  $m$ -w.f.f. string of  $S$ . Observe that  $\text{xnot } p_1$  is  $m + 1$ -w.f.f..

Let us consider  $S, p_1$ . Note that  $\text{xnot } p_1$  is w.f.f..

Let us consider  $S$ . One can verify that  $\text{TheEqSymbOf } S$  is non own.

Let us consider  $S, X$ . We say that  $X$  is  $S$ -mincover if and only if:

(Def. 34) For every  $p_1$  holds  $p_1 \in X$  iff  $\text{xnot } p_1 \notin X$ .

One can prove the following propositions:

(17)  $\text{Depth}(\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4) = 1 + \max(\text{Depth } p_3, \text{Depth } p_4)$  and  $\text{Depth}(\langle l \rangle \wedge p_3) = \text{Depth } p_3 + 1$ .

(18) If  $\text{Depth } p_1 = m + 1$ , then  $p_1$  is exal iff  $p_1 \in m\text{-ExFormulasOf } S$  and  $p_1$  is non exal iff  $p_1 \in m\text{-NorFormulasOf } S$ .

(19)  $I\text{-TruthEval}\langle l \rangle \wedge p_1 = \text{true}$  iff there exists  $u$  such that  $((l, u) \text{ReassignIn } I)\text{-TruthEval } p_1 = 1$  and  $I\text{-TruthEval}\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4 = \text{true}$  iff  $I\text{-TruthEval } p_3 = \text{false}$  and  $I\text{-TruthEval } p_4 = \text{false}$ .

In the sequel  $I$  denotes an  $(S, U)$ -interpreter-like function.

One can prove the following two propositions:

(20)  $(I, u)\text{-TermEval}(m + 1) \upharpoonright S\text{-termsOfMaxDepth}(m) = I\text{-TermEval} \upharpoonright S\text{-termsOfMaxDepth}(m)$ .

(21)  $I\text{-TermEval}(t) = I(S\text{-firstChar}(t))(I\text{-TermEval} \cdot \text{SubTerms } t)$ .

Let us consider  $S, p_1$ . The functor  $\text{SubWffsOf } p_1$  is defined as follows:

(Def. 35)(i) There exist  $p_3, p$  such that  $p$  is  $\text{AllSymbolsOf } S$ -valued and  $\text{SubWffsOf } p_1 = \langle p_3, p \rangle$  and  $p_1 = \langle S\text{-firstChar}(p_1) \rangle \wedge p_3 \wedge p$  if  $p_1$  is non 0-w.f.f.,

(ii)  $\text{SubWffsOf } p_1 = \langle p_1, \emptyset \rangle$ , otherwise.

Let us consider  $S, p_1$ . The functor  $\text{head } p_1$  yields a w.f.f. string of  $S$  and is defined as follows:

(Def. 36)  $\text{head } p_1 = (\text{SubWffsOf } p_1)_1$ .

The functor  $\text{tail } p_1$  yields an element of  $(\text{AllSymbolsOf } S)^*$  and is defined by:

(Def. 37)  $\text{tail } p_1 = (\text{SubWffsOf } p_1)_2$ .

Let us consider  $S, m$ . One can verify that  $(S\text{-formulasOfMaxDepth } m) \setminus \text{AllFormulasOf } S$  is empty.

Let us consider  $S$ . Observe that  $\text{AtomicFormulasOf } S \setminus \text{AllFormulasOf } S$  is empty.

We now state two propositions:

(22)  $\text{Depth}(\langle l \rangle \wedge p_3) > \text{Depth } p_3$  and  $\text{Depth}(\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4) > \text{Depth } p_3$  and  $\text{Depth}(\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4) > \text{Depth } p_4$ .

(23) If  $p_1$  is not 0-w.f.f., then  $p_1 = \langle x \rangle \wedge p_4 \wedge p_2$  iff  $x = S\text{-firstChar}(p_1)$  and  $p_4 = \text{head } p_1$  and  $p_2 = \text{tail } p_1$ .

Let us consider  $S, m_2$ . Observe that there exists a non 0-w.f.f.  $m_2$ -w.f.f. string of  $S$  which is non exal.

Let us consider  $S$  and let  $p_1$  be an exal w.f.f. string of  $S$ . One can verify that  $\text{tail } p_1$  is empty.

Let us consider  $S$  and let  $p_1$  be a non exal non 0-w.f.f. w.f.f. string of  $S$ . Then  $\text{tail } p_1$  is a w.f.f. string of  $S$ .

Let us consider  $S$  and let  $p_1$  be a non exal non 0-w.f.f. w.f.f. string of  $S$ . One can check that  $\text{tail } p_1$  is w.f.f..

Let us consider  $S$  and let  $p_1$  be a non 0-w.f.f. non exal w.f.f. string of  $S$ . One can verify that  $S\text{-firstChar}(p_1) \div \text{TheNorSymbOf } S$  is empty.

Let us consider  $m, S$  and let  $p_1$  be an  $m + 1$ -w.f.f. string of  $S$ . Note that  $\text{head } p_1$  is  $m$ -w.f.f..

Let us consider  $m, S$  and let  $p_1$  be an  $m + 1$ -w.f.f. non exal non 0-w.f.f. string of  $S$ . Observe that  $\text{tail } p_1$  is  $m$ -w.f.f..

One can prove the following proposition

(24) For every element  $I$  of  $U\text{-InterpretersOf } S$  holds  $(I, m)\text{-TruthEval} \in \text{Boolean}^{S\text{-formulasOfMaxDepth } m}$ .

Let us consider  $S$ . One can check that there exists an of-atomic-formula element of  $S$  which is non literal.

One can prove the following proposition

(25) If  $l_2 \notin \text{rng } p$ , then  $((l_2, u)\text{ReassignIn } I)\text{-TermEval}(p) = I\text{-TermEval}(p)$ .

Let us consider  $X, S, s$ . We say that  $s$  is  $X$ -occurring if and only if:

(Def. 38)  $s \in \text{SymbolsOf}(((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \cap X)$ .

Let us consider  $S, s$  and let us consider  $X$ . We say that  $X$  is  $s$ -containing if and only if:

(Def. 39)  $s \in \text{SymbolsOf}((\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}) \cap X$ .

Let us consider  $X, S, s$ . We introduce  $s$  is  $X$ -absent as an antonym of  $s$  is  $X$ -occurring.

Let us consider  $S, s, X$ . We introduce  $X$  is  $s$ -free as an antonym of  $X$  is  $s$ -containing.

Let  $X$  be a finite set and let us consider  $S$ . Observe that there exists a literal element of  $S$  which is  $X$ -absent.

Let us consider  $S, t$ . Note that  $\text{rng } t \cap \text{LettersOf } S$  is non empty.

Let us consider  $S, p_1$ . One can verify that  $\text{rng } p_1 \cap \text{LettersOf } S$  is non empty.

Let us consider  $B, S$  and let  $A$  be a subset of  $B$ . Note that every element of  $S$  which is  $A$ -occurring is also  $B$ -occurring.

Let us consider  $A, B, S$ . Observe that every element of  $S$  which is  $A$  null  $B$ -absent is also  $A \cap B$ -absent.

Let  $F$  be a finite set and let us consider  $A, S$ . Note that every  $F$ -absent element of  $S$  which is  $A$ -absent is also  $A \cup F$ -absent.

Let us consider  $S, U$  and let  $I$  be an  $(S, U)$ -interpreter-like function. One can check that  $\text{OwnSymbolsOf } S \setminus \text{dom } I$  is empty.

One can prove the following proposition

(26) There exists  $u$  such that  $u = I(l)(\emptyset)$  and  $(l, u) \text{ ReassignIn } I = I$ .

Let us consider  $S, X$ . We say that  $X$  is  $S$ -covering if and only if:

(Def. 40) For every  $p_1$  holds  $p_1 \in X$  or  $\text{xnot } p_1 \in X$ .

Let us consider  $S$ . One can check that every set which is  $S$ -mincover is also  $S$ -covering.

Let us consider  $U$ , let  $p_1$  be a non 0-w.f.f. non exal w.f.f. string of  $S$ , and let  $I$  be an element of  $U\text{-InterpretersOf } S$ .

One can verify that  $(I\text{-TruthEval } p_1) \div ((I\text{-TruthEval head } p_1) \text{ 'nor' } (I\text{-TruthEval tail } p_1))$  is empty.

The functor  $\text{ExFormulasOf } S$  yielding a subset of  $(\text{AllSymbolsOf } S)^* \setminus \{\emptyset\}$  is defined by:

(Def. 41)  $\text{ExFormulasOf } S = \{p_1; p_1 \text{ ranges over strings of } S: p_1 \text{ is w.f.f. } \wedge p_1 \text{ is exal}\}$ .

Let us consider  $S$ . Note that  $\text{ExFormulasOf } S$  is non empty.

Let us consider  $S$ . One can check that every element of  $\text{ExFormulasOf } S$  is exal and w.f.f..

Let us consider  $S$ . Note that every element of  $\text{ExFormulasOf } S$  is w.f.f..

Let us consider  $S$ . Observe that every element of  $\text{ExFormulasOf } S$  is exal.

Let us consider  $S$ . Observe that  $\text{ExFormulasOf } S \setminus \text{AllFormulasOf } S$  is empty.

Let us consider  $U, S_1$  and let  $S_2$  be an  $S_1$ -extending language. Note that every function which is  $(S_2, U)$ -interpreter-like is also  $(S_1, U)$ -interpreter-like.

Let us consider  $U, S_1$ , let  $S_2$  be an  $S_1$ -extending language, and let  $I$  be an  $(S_2, U)$ -interpreter-like function. Observe that  $I \upharpoonright \text{OwnSymbolsOf } S_1$  is  $(S_1, U)$ -interpreter-like.

Let us consider  $U, S_1$ , let  $S_2$  be an  $S_1$ -extending language, let  $I_1$  be an element of  $U\text{-InterpretersOf } S_1$ , and let  $I_2$  be an  $(S_2, U)$ -interpreter-like function. Note that  $I_2 + \cdot I_1$  is  $(S_2, U)$ -interpreter-like.

Let us consider  $U, S$ , let  $I$  be an element of  $U\text{-InterpretersOf } S$ , and let us consider  $X$ . We say that  $X$  is  $I$ -satisfied if and only if:

(Def. 42) For every  $p_1$  such that  $p_1 \in X$  holds  $I\text{-TruthEval } p_1 = 1$ .

Let us consider  $S, U, X$  and let  $I$  be an element of  $U\text{-InterpretersOf } S$ . We say that  $I$  satisfies  $X$  if and only if:

(Def. 43)  $X$  is  $I$ -satisfied.

Let us consider  $U, S$ , let  $e$  be an empty set, and let  $I$  be an element of  $U\text{-InterpretersOf } S$ . Observe that  $e \text{ null } I$  is  $I$ -satisfied.

Let us consider  $X, U, S$  and let  $I$  be an element of  $U$ -InterpretersOf  $S$ . Observe that there exists a subset of  $X$  which is  $I$ -satisfied.

Let us consider  $U, S$  and let  $I$  be an element of  $U$ -InterpretersOf  $S$ . One can check that there exists a set which is  $I$ -satisfied.

Let us consider  $U, S$ , let  $I$  be an element of  $U$ -InterpretersOf  $S$ , and let  $X$  be an  $I$ -satisfied set. One can check that every subset of  $X$  is  $I$ -satisfied.

Let us consider  $U, S$ , let  $I$  be an element of  $U$ -InterpretersOf  $S$ , and let  $X, Y$  be  $I$ -satisfied sets. One can verify that  $X \cup Y$  is  $I$ -satisfied.

Let us consider  $U, S$ , let  $I$  be an element of  $U$ -InterpretersOf  $S$ , and let  $X$  be an  $I$ -satisfied set. Observe that  $I$  null  $X$  satisfies  $X$ .

Let us consider  $S, X$ . We say that  $X$  is  $S$ -correct if and only if the condition (Def. 44) is satisfied.

(Def. 44) Let  $U$  be a non empty set,  $I$  be an element of  $U$ -InterpretersOf  $S$ ,  $x$  be an  $I$ -satisfied set, and given  $p_1$ . If  $\langle x, p_1 \rangle \in X$ , then  $I$ -TruthEval  $p_1 = 1$ .

Let us consider  $S$ . Note that  $\emptyset$  null  $S$  is  $S$ -correct.

Let us consider  $S, X$ . Observe that there exists a subset of  $X$  which is  $S$ -correct.

Next we state two propositions:

(27) For every element  $I$  of  $U$ -InterpretersOf  $S$  holds  $I$ -TruthEval  $p_1 = 1$  iff  $\{p_1\}$  is  $I$ -satisfied.

(28)  $s$  is  $\{w\}$ -occurring iff  $s \in \text{rng } w$ .

Let us consider  $U, S$ , let us consider  $p_3, p_4$ , and let  $I$  be an element of  $U$ -InterpretersOf  $S$ . Observe that  $(I$ -TruthEval  $\langle \text{TheNorSymbOf } S \rangle \wedge p_3 \wedge p_4) \div ((I$ -TruthEval  $p_3) \text{ 'nor' } (I$ -TruthEval  $p_4))$  is empty.

Let us consider  $S, p_1, U$  and let  $I$  be an element of  $U$ -InterpretersOf  $S$ . Note that  $(I$ -TruthEval  $x$  not  $p_1) \div \neg(I$ -TruthEval  $p_1)$  is empty.

Let us consider  $X, S, p_1$ . We say that  $p_1$  is  $X$ -implied if and only if:

(Def. 45) For every non empty set  $U$  and for every element  $I$  of  $U$ -InterpretersOf  $S$  such that  $X$  is  $I$ -satisfied holds  $I$ -TruthEval  $p_1 = 1$ .

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