Continuity of Barycentric Coordinates in Euclidean Topological Spaces

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Summary. In this paper we present selected properties of barycentric coordinates in the Euclidean topological space. We prove the topological correspondence between a subset of an affine closed space of $\mathbb{E}^n$ and the set of vectors created from barycentric coordinates of points of this subset.

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The terminology and notation used here have been introduced in the following articles: [1], [3], [15], [25], [13], [18], [5], [4], [6], [12], [7], [8], [33], [21], [24], [2], [22], [20], [17], [30], [31], [23], [10], [26], [11], [16], [29], [14], [19], [27], [32], and [9].

1. Preliminaries

For simplicity, we adopt the following rules: $x$ denotes a set, $n, m, k$ denote natural numbers, $r$ denotes a real number, $V$ denotes a real linear space, $v, w$ denote vectors of $V$, $A_1$ denotes a finite subset of $V$, and $A_2$ denotes a finite affinely independent subset of $V$.

One can prove the following propositions:

(1) For all real-valued finite sequences $f_1, f_2$ and for every real number $r$ holds $(\text{Intervals}(f_1, r)) \triangleq \text{Intervals}(f_2, r) = \text{Intervals}(f_1 \triangleq f_2, r)$.

(2) Let $f_1, f_2$ be finite sequences. Then $x \in \prod(f_1 \triangle f_2)$ if and only if there exist finite sequences $p_1, p_2$ such that $x = p_1 \triangle p_2$ and $p_1 \in \prod f_1$ and $p_2 \in \prod f_2$. 
(3) $V$ is finite dimensional iff $\Omega_V$ is finite dimensional.

Let $V$ be a finite dimensional real linear space. One can verify that every affinely independent subset of $V$ is finite.

Let us consider $n$. One can check that $E^n_T$ is add-continuous and mult-continuous and $E^n_T$ is finite dimensional.

In the sequel $p_3$ denotes a point of $E^n_T$, $A_3$ denotes a subset of $E^n_T$, and $A_5$ denotes a subset of $E_k^n$.

Next we state three propositions:

(4) $\dim(E^n_T) = n$.

(5) Let $V$ be a finite dimensional real linear space and $A$ be an affinely independent subset of $V$. Then $\overline{A} \leq 1 + \dim(V)$.

(6) Let $V$ be a finite dimensional real linear space and $A$ be an affinely independent subset of $V$. Then $A = \dim(V)+1$ if and only if $\text{Affin} A = \Omega_V$.

2. Open and Closed Subsets of a Subspace of the Euclidean Topological Space

One can prove the following propositions:

(7) If $k \leq n$ and $A_3 = \{v \in E^n_T: v|k \in A_3\}$, then $A_3$ is open iff $A_5$ is open.

(8) Let $A$ be a subset of $E^{k+n}_T$. Suppose $A = \{v \cap (n \mapsto 0): v \text{ ranges over elements of } E^k_T\}$. Let $B$ be a subset of $E^{k+n}_T[A]$. Suppose $B = \{v; v \text{ ranges over points of } E^{k+n}_T: v|k \in A_5 \land v \in A\}$. Then $A_5$ is open if and only if $B$ is open.

(9) For every affinely independent subset $A$ of $V$ and for every subset $B$ of $V$ such that $B \subseteq A$ holds $\text{conv} A \cap \text{Affin} B = \text{conv} B$.

(10) Let $V$ be a non empty RLS structure, $A$ be a non empty set, $f$ be a partial function from $A$ to the carrier of $V$, and $X$ be a set. Then $(r \cdot f)^X = r \cdot f^X$.

(11) If $0, \ldots, 0 \in A_3$, then $\text{Affin} A_3 = \Omega_{\text{Lin}(A_3)}$.

Let $V$ be a non empty additive loop structure, let $A$ be a finite subset of $V$, and let $v$ be an element of $V$. Note that $v + A$ is finite.

Let $V$ be a non empty RLS structure, let $A$ be a finite subset of $V$, and let us consider $r$. Observe that $r \cdot A$ is finite.

Next we state the proposition

(12) For every subset $A$ of $V$ holds $\overline{A} = r \cdot \overline{A}$ iff $r \neq 0$ or $A$ is trivial.

Let $V$ be a non empty RLS structure, let $f$ be a finite sequence of elements of $V$, and let us consider $r$. Note that $r \cdot f$ is finite sequence-like.
3. The Vector of Barycentric Coordinates

Let $X$ be a finite set. A one-to-one finite sequence is said to be an enumeration of $X$ if:

(Def. 1) $\text{rng } x = X$.

Let $X$ be a 1-sorted structure and let $A$ be a finite subset of $X$. We see that the enumeration of $A$ is a one-to-one finite sequence of elements of $X$.

In the sequel $E_1$ denotes an enumeration of $A_2$ and $E_2$ denotes an enumeration of $A_4$.

One can prove the following three propositions:

(13) Let $V$ be an Abelian add-associative right zeroed right complementable non empty additive loop structure, $A$ be a finite subset of $V$, $E$ be an enumeration of $A$, and $v$ be an element of $V$. Then $E + A \rightarrow v$ is an enumeration of $v + A$.

(14) For every enumeration $E$ of $A_1$ holds $r \cdot E$ is an enumeration of $r \cdot A_1$ if $r \neq 0$ or $A_1$ is trivial.

(15) Let $M$ be a matrix over $\mathbb{R}_F$ of dimension $n \times m$. Suppose $\text{rk}(M) = n$. Let $C$ be a finite subset of $E_1$ and $E$ be an enumeration of $A$. Then $M \cdot E$ is an enumeration of $(M \cdot E^n)^C_A$.

Let us consider $V$, $A_1$, let $E$ be an enumeration of $A_1$, and let us consider $x$. The functor $x \rightarrow E$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 2) $x \rightarrow E = (x \rightarrow A_1) \cdot E$.

The following propositions are true:

(16) For every enumeration $E$ of $A_1$ holds $\text{len}(x \rightarrow E) = A_1$.

(17) For every enumeration $E$ of $v + A_2$ such that $w \in \text{Affin} A_2$ and $E = E_1 + A_2 \rightarrow v$ holds $w \rightarrow E_1 = v + w \rightarrow E$.

(18) For every enumeration $r_1$ of $r \cdot A_2$ such that $v \in \text{Affin} A_2$ and $r_1 = r \cdot E_1$ and $r \neq 0$ holds $v \rightarrow E_1 = r \cdot v \rightarrow r_1$.

(19) Let $M$ be a matrix over $\mathbb{R}_F$ of dimension $n \times m$. Suppose $\text{rk}(M) = n$. Let $M_1$ be an enumeration of $(M \cdot E_1)^C_A$. If $M_1 = M \cdot E_2$, then for every $p_3$ such that $p_3 \in \text{Affin} A_4$ holds $p_3 \rightarrow E_2 = (M_1)(p_3) \rightarrow M_1$.

(20) Let $A$ be a subset of $V$. Suppose $A \subseteq A_2$ and $x \in \text{Affin} A_2$. Then $x \in \text{Affin} A$ if and only if for every set $y$ such that $y \in \text{dom}(x \rightarrow E_1)$ and $E_1(y) \notin A$ holds $(x \rightarrow E_1)(y) = 0$.

(21) For every $E_1$ such that $x \in \text{Affin} A_2$ holds $x \in \text{Affin}(E_1 \cap \text{Seg} k)$ iff $x \rightarrow E_1 = ((x \rightarrow E_1)(k) \cap \overline{(A_2 - k) \rightarrow 0})$.

(22) For every $E_1$ such that $k \leq \overline{A_2}$ and $x \in \text{Affin} A_2$ holds $x \in \text{Affin}(A_2 \setminus E_1 \cap \text{Seg} k)$ iff $x \rightarrow E_1 = (k \rightarrow 0) \cap ((x \rightarrow E_1)_{|k})$. 

(23) Suppose \((0,\ldots,0)\) ∈ \(A_4\) and \(E_2(\text{len } E_2) = (0,\ldots,0)\). Then
(i) \(\text{rng}(E_2([\overline{A_4} -' 1])) = A_4 \setminus \{(0,\ldots,0)\}\), and
(ii) for every subset \(A\) of the \(n\)-dimension vector space over \(\mathbb{R}_F\) such that \(A_4 = A\) holds \(E_2([\overline{A_4} -' 1])\) is an ordered basis of \(\text{Lin}(A)\).
(24) Let \(A\) be a subset of the \(n\)-dimension vector space over \(\mathbb{R}_F\). Suppose \(A_4 = A\) and \((0,\ldots,0)\) ∈ \(A_4\) and \(E_2(\text{len } E_2) = (0,\ldots,0)\). Let \(B\) be an ordered basis of \(\text{Lin}(A)\). If \(B = E_2([\overline{A_4} -' 1])\), then for every element \(v\) of \(\text{Lin}(A)\) holds \(v \rightarrow B = (v \rightarrow E_2)([\overline{A_4} -' 1])\).
(25) For all \(E_2, A_3\) such that \(k \leq n\) and \(\overline{A_4} = n + 1\) and \(A_3 = \{p_3 : (p_3 \rightarrow E_2) | k \in A_5\}\) holds \(A_5\) is open iff \(A_3\) is open.
(26) For every \(E_2\) such that \(k \leq n\) and \(\overline{A_4} = n + 1\) and \(A_3 = \{p_3 : (p_3 \rightarrow E_2) | k \in A_5\}\) holds \(A_5\) is closed iff \(A_3\) is closed.
Let us consider \(n\). One can verify that every subset of \(\mathcal{E}_4^n\) which is affine is also closed.

In the sequel \(p_4\) denotes an element of \(\mathcal{E}_4^n | \text{Affin } A_4\).
Next we state two propositions:
(27) For every \(E_2\) and for every subset \(B\) of \(\mathcal{E}_4^n | \text{Affin } A_4\) such that \(k < \overline{A_4}\) and \(B = \{p_4 : (p_4 \rightarrow E_2) | k \in A_5\}\) holds \(A_5\) is open iff \(B\) is open.
(28) Let given \(E_2\) and \(B\) be a subset of \(\mathcal{E}_4^n | \text{Affin } A_4\). Suppose \(k < \overline{A_4}\) and \(B = \{p_4 : (p_4 \rightarrow E_2) | k \in A_5\}\). Then \(A_5\) is closed if and only if \(B\) is closed.
Let us consider \(n\) and let \(p, q\) be points of \(\mathcal{E}_4^n\). Observe that halfline\((p, q)\) is closed.

4. Continuity of Barycentric Coordinates

Let us consider \(V\), let \(A\) be a subset of \(V\), and let us consider \(x\). The functor \(\vdash (A, x)\) yielding a function from \(V\) into \(\mathbb{R}^1\) is defined as follows:
(Def. 3) \((\vdash (A, x))(v) = (v \rightarrow A)(x)\).
One can prove the following four propositions:
(29) For every subset \(A\) of \(V\) such that \(x \notin A\) holds \(\vdash (A, x) = \Omega_V \iff 0\).
(30) For every affinely independent subset \(A\) of \(V\) such that \(\vdash (A, x) = \Omega_V \iff 0\) holds \(x \notin A\).
(31) \(\vdash (A_4, x) | \text{Affin } A_4\) is a continuous function from \(\mathcal{E}_4^n | \text{Affin } A_4\) into \(\mathbb{R}^1\).
(32) If \(\overline{A_4} = n + 1\), then \(\vdash (A_4, x)\) is continuous.
Let us consider \(n, A_4\). Note that \(\text{conv } A_4\) is closed.
We now state the proposition
If $A_4 = n + 1$, then $\text{Int} A_4$ is open.

References


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