Cayley’s Theorem

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Summary. The article formalizes the Cayley’s theorem saying that every group $G$ is isomorphic to a subgroup of the symmetric group on $G$.

The notation and terminology used in this paper have been introduced in the following papers: [3], [6], [4], [5], [10], [11], [7], [2], [1], [9], and [8].

In this paper $X$, $Y$ denote sets, $G$ denotes a group, and $n$ denotes a natural number.

Let us consider $X$. Note that $\emptyset, X$ is onto.

Let us observe that every set which is permutational is also functional.

Let us consider $X$. The functor permutations $X$ is defined as follows:

(Def. 1) permutations $X = \{ f : f \text{ ranges over permutations of } X \}$.

Next we state three propositions:

(1) For every set $f$ such that $f \in \text{permutations } X$ holds $f$ is a permutation of $X$.

(2) permutations $X \subseteq X^X$.

(3) permutations $\text{Seg } n = \text{the permutations of } n$.

Let us consider $X$. One can verify that permutations $X$ is non empty and functional.

Let $X$ be a finite set. One can verify that permutations $X$ is finite.

Next we state the proposition

(4) permutations $\emptyset = 1$.

Let us consider $X$. The functor SymGroup $X$ yields a strict constituted functions multiplicative magma and is defined by:
The carrier of SymGroup $X = \text{permutations } X$ and for all elements $x, y$ of SymGroup $X$ holds $x \cdot y = (y \text{ qua function}) \cdot x$.

One can prove the following proposition

(5) Every element of SymGroup $X$ is a permutation of $X$.

Let us consider $X$. Note that SymGroup $X$ is non empty, associative, and group-like.

The following propositions are true:

(6) $1_{\text{SymGroup } X} = \text{id}_X$.

(7) For every element $x$ of SymGroup $X$ holds $x^{-1} = (x \text{ qua function})^{-1}$.

Let us consider $n$. One can verify that $A_n$ is constituted functions.

One can prove the following proposition

(8) SymGroup $\text{Seg } n = A_n$.

Let $X$ be a finite set. Observe that SymGroup $X$ is finite.

We now state the proposition

(9) SymGroup $\emptyset = \text{Trivial-multMagma}$.

Let us note that SymGroup $\emptyset$ is trivial.

Let us consider $X, Y$ and let $p$ be a function from $X$ into $Y$. Let us assume that $X \neq \emptyset$ and $Y \neq \emptyset$ and $p$ is bijective. The functor SymGroupsIso $p$ yielding a function from SymGroup $X$ into SymGroup $Y$ is defined by:

(Def. 3) For every element $x$ of SymGroup $X$ holds $(\text{SymGroupsIso } p)(x) = p \cdot x \cdot p^{-1}$.

We now state four propositions:

(10) For all non empty sets $X, Y$ and for every function $p$ from $X$ into $Y$ such that $p$ is bijective holds SymGroupsIso $p$ is multiplicative.

(11) For all non empty sets $X, Y$ and for every function $p$ from $X$ into $Y$ such that $p$ is bijective holds SymGroupsIso $p$ is one-to-one.

(12) For all non empty sets $X, Y$ and for every function $p$ from $X$ into $Y$ such that $p$ is bijective holds SymGroupsIso $p$ is onto.

(13) If $X \approx Y$, then SymGroup $X$ and SymGroup $Y$ are isomorphic.

Let us consider $G$. The functor CayleyIso $G$ yields a function from $G$ into SymGroup (the carrier of $G$) and is defined as follows:

(Def. 4) For every element $g$ of $G$ holds $(\text{CayleyIso } G)(g) = \cdot g$.

Let us consider $G$. One can verify that CayleyIso $G$ is multiplicative.

Let us consider $G$. One can verify that CayleyIso $G$ is one-to-one.

One can prove the following proposition

(14) $G$ and $\text{Im CayleyIso } G$ are isomorphic.
REFERENCES


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