Representation Theorem for Stacks

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Summary. In the paper the concept of stacks is formalized. As the main result the Theorem of Representation for Stacks is given. Formalization is done according to [13].

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The papers [6], [15], [14], [2], [4], [7], [16], [8], [9], [10], [5], [1], [17], [11], [19], [21], [20], [3], [18], and [12] provide the terminology and notation for this paper.

1. INTRODUCTIONS

In this paper \( i \) is a natural number and \( x \) is a set.

Let \( A \) be a set and let \( s_1, s_2 \) be finite sequences of elements of \( A \). Then \( s_1 \circ s_2 \) is an element of \( A^* \).

Let \( A \) be a set, let \( i \) be a natural number, and let \( s \) be a finite sequence of elements of \( A \). Then \( s|_i \) is an element of \( A^* \).

The following two propositions are true:

1. \( \emptyset|_i = \emptyset \).
2. Let \( D \) be a non empty set and \( s \) be a finite sequence of elements of \( D \). Suppose \( s \neq \emptyset \). Then there exists a finite sequence \( w \) of elements of \( D \) and there exists an element \( n \) of \( D \) such that \( s = \langle n \rangle \circ w \).

The scheme \( \text{IndSeqD} \) deals with a non empty set \( A \) and a unary predicate \( P \), and states that:

For every finite sequence \( p \) of elements of \( A \) holds \( P[p] \) provided the following conditions are met:

- \( P[\varepsilon_A] \), and
• For every finite sequence \( p \) of elements of \( \mathcal{A} \) and for every element 
\( x \) of \( \mathcal{A} \) such that \( \mathcal{P}[p] \) holds \( \mathcal{P}[\langle x \rangle] \).

Let \( C, D \) be non empty sets and let \( R \) be a binary relation. A function from 
\( C \times D \) into \( D \) is said to be a binary operation of \( C \) and \( D \) being congruence 
w.r.t. \( R \) if:

(Def. 1) For every element \( x \) of \( C \) and for all elements \( y_1, y_2 \) of \( D \) such that 
\( \langle \langle y_1, y_2 \rangle \rangle \in R \) holds 
\( \langle \langle \text{it}(x, y_1), \text{it}(x, y_2) \rangle \rangle \in R \).

The scheme LambdaD2 deals with non empty sets \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) and a binary 
functor \( \mathcal{F} \) yielding an element of \( \mathcal{C} \), and states that:

There exists a function \( M \) from \( \mathcal{A} \times \mathcal{B} \) into \( \mathcal{C} \) such that for every 
element \( i \) of \( \mathcal{A} \) and for every element \( j \) of \( \mathcal{B} \) holds 
\( M(i, j) = \mathcal{F}(i, j) \) for all values of the parameters.

Let \( C, D \) be non empty sets, let \( R \) be an equivalence relation of \( D \), and let 
\( b \) be a function from \( C \times D \) into \( D \). Let us assume that \( b \) is a binary operation of 
\( C \) and \( D \) being congruence w.r.t. \( R \). The functor \( b/_{\mathcal{R}} \) yielding a function from 
\( C \times \text{Classes } \mathcal{R} \) into \( \text{Classes } \mathcal{R} \) is defined as follows:

(Def. 2) For all sets \( x, y, y_1 \) such that \( x \in C \) and \( y \in \text{Classes } \mathcal{R} \) and \( y_1 \in y \) holds 
\( b/_{\mathcal{R}}(x, y) = [b(x, y_1)]_{\mathcal{R}} \).

Let \( A, B \) be non empty sets, let \( C \) be a subset of \( A \), let \( D \) be a subset of \( B \), 
let \( f \) be a function from \( A \) into \( B \), and let \( g \) be a function from \( C \) into \( D \). Then 
\( f + g \) is a function from \( A \) into \( B \).

### 2. Stack Algebra

We introduce stack systems which are extensions of 2-sorted and are systems
\( \langle \text{a carrier, a carrier'}, empty stacks, a push function, a pop function, a top function } \rangle \),
where the carrier is a set, the carrier’ is a set, the empty stacks constitute 
subsets of the carrier’, the push function is a function from the carrier\( \times \)the 
carrier’ into the carrier’, the pop function is a function from the carrier’ into the 
carrier’, and the top function is a function from the carrier’ into the carrier.

Let \( a_1 \) be a non empty set, let \( a_2 \) be a set, let \( a_3 \) be a subset of \( a_2 \), let \( a_4 \) be 
a function from \( a_1 \times a_2 \) into \( a_2 \), let \( a_5 \) be a function from \( a_2 \) into \( a_2 \), and let \( a_6 \) 
be a function from \( a_2 \) into \( a_1 \). Observe that stack system\( \langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle \) is 
non empty.

Let \( a_1 \) be a set, let \( a_2 \) be a non empty set, let \( a_3 \) be a subset of \( a_2 \), let \( a_4 \) be 
a function from \( a_1 \times a_2 \) into \( a_2 \), let \( a_5 \) be a function from \( a_2 \) into \( a_2 \), and let \( a_6 \) be a 
function from \( a_2 \) into \( a_1 \). One can verify that stack system\( \langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle \) 
is non void.

Let us note that there exists a stack system which is non empty, non void, 
and strict.
Let $X$ be a stack system. A stack of $X$ is an element of the carrier’ of $X$.

Let $X$ be a non empty non void stack system and let $s$ be a stack of $X$. The predicate $\text{empty}(s)$ is defined by:

(Def. 3) $s \in \text{the empty stacks of } X$.

The functor $\text{pop } s$ yields a stack of $X$ and is defined by:

(Def. 4) $\text{pop } s = (\text{the pop function of } X)(s)$.

The functor $\text{top } s$ yields an element of $X$ and is defined by:

(Def. 5) $\text{top } s = (\text{the top function of } X)(s)$.

Let $e$ be an element of $X$. The functor $\text{push}(e, s)$ yields a stack of $X$ and is defined by:

(Def. 6) $\text{push}(e, s) = (\text{the push function of } X)(e, s)$.

Let $A$ be a non empty set. Standard stack system over $A$ yielding a non empty non void strict stack system is defined by the conditions (Def. 7).

(Def. 7)(i) The carrier of standard stack system over $A = A$,

(ii) the carrier’ of standard stack system over $A = A^*$, and

(iii) for every stack $s$ of standard stack system over $A$ holds $\text{empty}(s)$

if $s$ is empty and for every finite sequence $g$ such that $g = s$ holds if
not $\text{empty}(s)$, then $\text{top } s = g(1)$ and $\text{pop } s = g\mid_{1}$ and if $\text{empty}(s)$, then
$\text{top } s = \text{the element of standard stack system over } A$ and $\text{pop } s = \emptyset$ and for
every element $e$ of standard stack system over $A$ holds $\text{push}(e, s) = \langle e \rangle \upharpoonright g$.

In the sequel $A$ denotes a non empty set, $c$ denotes an element of standard stack system over $A$, and $m$ denotes a stack of standard stack system over $A$.

Let us consider $A$. Note that every stack of standard stack system over $A$ is relation-like and function-like.

Let us consider $A$. Observe that every stack of standard stack system over $A$ is finite sequence-like.

We adopt the following convention: $X$ denotes a non empty non void stack system, $s, s_1$ denote stacks of $X$, and $e, e_1, e_2$ denote elements of $X$.

Let us consider $X$. We say that $X$ is pop-finite if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let $f$ be a function from $\mathbb{N}$ into the carrier’ of $X$. Then there exists a
natural number $i$ and there exists $s$ such that $f(i) = s$ and if not $\text{empty}(s)$,
then $f(i + 1) \neq \text{pop } s$.

We say that $X$ is push-pop if and only if:

(Def. 9) If not $\text{empty}(s)$, then $s = \text{push}(\text{top } s, \text{pop } s)$.

We say that $X$ is top-push if and only if:

(Def. 10) $e = \text{top } \text{push}(e, s)$.

We say that $X$ is pop-push if and only if:

(Def. 11) $s = \text{pop } \text{push}(e, s)$.
We say that $X$ is push-non-empty if and only if:

(Def. 12) not empty(push($e, s$)).

Let $A$ be a non empty set. One can verify the following observations:

* standard stack system over $A$ is pop-finite,
* standard stack system over $A$ is push-pop,
* standard stack system over $A$ is top-push,
* standard stack system over $A$ is pop-push, and
* standard stack system over $A$ is push-non-empty.

Let us observe that there exists a non empty non void stack system which is pop-finite, push-pop, top-push, pop-push, push-non-empty, and strict.

A stack algebra is a pop-finite push-pop top-push pop-push push-non-empty non empty non void stack system.

Next we state the proposition

(3) For every non empty non void stack system $X$ such that $X$ is pop-finite there exists a stack $s$ of $X$ such that empty($s$).

Let $X$ be a pop-finite non empty non void stack system. Note that the empty stacks of $X$ is non empty.

We now state two propositions:

(4) If $X$ is top-push and pop-push and push($e_1, s_1$) = push($e_2, s_2$), then $e_1 = e_2$ and $s_1 = s_2$.

(5) If $X$ is push-pop and not empty($s_1$) and not empty($s_2$) and pop $s_1 = pop s_2$ and top $s_1 = top s_2$, then $s_1 = s_2$.

3. SCHEMES OF INDUCTION

Now we present three schemes. The scheme $INDsch$ deals with a stack algebra $A$, a stack $B$ of $A$, and a unary predicate $P$, and states that:

$P[B]$

provided the following conditions are satisfied:

- For every stack $s$ of $A$ such that empty($s$) holds $P[s]$, and
- For every stack $s$ of $A$ and for every element $e$ of $A$ such that $P[s]$ holds $P[push(e, s)]$.

The scheme $EXsch$ deals with a stack algebra $A$, a stack $B$ of $A$, a non empty set $C$, an element $D$ of $C$, and a binary functor $F$ yielding an element of $C$, and states that:

There exists an element $a$ of $C$ and there exists a function $F$ from the carrier’ of $A$ into $C$ such that

(i) $a = F(B)$,
(ii) for every stack $s_1$ of $A$ such that empty($s_1$) holds $F(s_1) = D$, and
(iii) for every stack $s_1$ of $A$ and for every element $e$ of $A$ holds
$F(push(e, s_1)) = F(e, F(s_1))$
for all values of the parameters.

The scheme UNIQsch deals with a stack algebra $A$, a stack $B$ of $A$, a non
empty set $C$, an element $D$ of $C$, and a binary functor $F$ yielding an element of
$C$, and states that:

Let $a_1, a_2$ be elements of $C$. Suppose that

(i) there exists a function $F$ from the carrier' of $A$ into $C$ such that
$F(s_1) = D$ and for every stack $s_1$ of $A$ and for every element $e$ of $A$ holds
$F(push(e, s_1)) = F(e, F(s_1))$, and

(ii) there exists a function $F$ from the carrier' of $A$ into $C$ such that
$F(s_1) = D$ and for every stack $s_1$ of $A$ and for every element $e$ of $A$ holds
$F(push(e, s_1)) = F(e, F(s_1))$.

Then $a_1 = a_2$
for all values of the parameters.

4. Stack Congruence

We adopt the following rules: $X$ is a stack algebra, $s, s_1, s_2, s_3$ are stacks of
$X$, and $e, e_1, e_2, e_3$ are elements of $X$.

Let us consider $X, s$. The functor $|s|$ yielding an element of
(the carrier of $X$)$^*$ is defined by the condition (Def. 13).

(Def. 13) There exists a function $F$ from the carrier' of $X$ into (the carrier of $X$)$^*$
such that $|s| = F(s)$ and for every $s_1$ such that empty($s_1$) holds $F(s_1) = \emptyset$
and for all $s_1, e$ holds $F(push(e, s_1)) = \langle e \rangle \triangleleft F(s_1)$.

Next we state several propositions:

(6) If empty($s$), then $|s| = \emptyset$.
(7) If not empty($s$), then $|s| = (top s) \triangleleft |pop s|$.
(8) If not empty($s$), then $|pop s| = |s|_{11}$.
(9) $|push(e, s)| = \langle e \rangle \triangleleft |s|$.
(10) If not empty($s$), then top $s = |s|(1)$.
(11) If $|s| = \emptyset$, then empty($s$).
(12) For every stack $s$ of standard stack system over $A$ holds $|s| = s$.
(13) For every element $x$ of (the carrier of $X$)$^*$ there exists $s$ such that $|s| = x$.

Let us consider $X, s_1, s_2$. The predicate $s_1 =_G s_2$ is defined as follows:

(Def. 14) $|s_1| = |s_2|$.

Let us notice that the predicate $s_1 =_G s_2$ is reflexive and symmetric.

The following propositions are true:
(14) If $s_1 = G s_2$ and $s_2 = G s_3$, then $s_1 = G s_3$.
(15) If $s_1 = G s_2$ and empty($s_1$), then empty($s_2$).
(16) If empty($s_1$) and empty($s_2$), then $s_1 = G s_2$.
(17) If $s_1 = G s_2$, then push($e, s_1$) = $G$ push($e, s_2$).
(18) If $s_1 = G s_2$ and not empty($s_1$), then pop $s_1 = G$ pop $s_2$.
(19) If $s_1 = G s_2$ and not empty($s_1$), then top $s_1 = $ top $s_2$.

Let us consider $X$. We say that $X$ is proper for identity if and only if:

(Def. 15) For all $s_1, s_2$ such that $s_1 = G s_2$ holds $s_1 = s_2$.

Let us consider $A$. Observe that standard stack system over $A$ is proper for identity.

Let us consider $X$. The functor $==_X$ yields a binary relation on the carrier’ of $X$ and is defined as follows:

(Def. 16) $\langle s_1, s_2 \rangle \in ==_X$ iff $s_1 = G s_2$.

Let us consider $X$. Observe that $==_X$ is total, symmetric, and transitive.

One can prove the following proposition

(20) If empty($s$), then $[s] ==_X = $ the empty stacks of $X$.

Let us consider $X$, $s$. The functor coset $s$ yielding a subset of the carrier’ of $X$ is defined by the conditions (Def. 17).

(Def. 17)(i) $s \in$ coset $s$,
(ii) for all $e, s_1$ such that $s_1 \in$ coset $s$ holds push($e, s_1$) $\in$ coset $s$ and if not empty($s_1$), then pop $s_1 \in$ coset $s$, and
(iii) for every subset $A$ of the carrier’ of $X$ such that $s \in A$ and for all $e, s_1$ such that $s_1 \in A$ holds push($e, s_1$) $\in A$ and if not empty($s_1$), then pop $s_1 \in A$ holds coset $s \subseteq A$.

Next we state three propositions:

(21) If push($e, s$) $\in$ coset $s_1$, then $s \in$ coset $s_1$ and if not empty($s$) and pop $s \in$ coset $s_1$, then $s \in$ coset $s_1$.
(22) $s \in$ coset push($e, s$) and if not empty($s$), then $s \in$ coset pop $s$.
(23) There exists $s_1$ such that empty($s_1$) and $s_1 \in$ coset $s$.

Let us consider $A$ and let $R$ be a binary relation on $A$. Note that there exists a reduction sequence w.r.t. $R$ which is $A$-valued.

Let us consider $X$. The construction reduction $X$ yielding a binary relation on the carrier’ of $X$ is defined as follows:

(Def. 18) $\langle s_1, s_2 \rangle \in$ the construction reduction $X$ iff not empty($s_1$) and $s_2 = \ $ pop $s_1$ or there exists $e$ such that $s_2 = $ push($e, s_1$).

Next we state the proposition

(24) Let $R$ be a binary relation on $A$ and $t$ be a reduction sequence w.r.t. $R$. Then $t(1) \in A$ if and only if $t$ is $A$-valued.
The scheme *PathIND* deals with a non-empty set $A$, elements $B, C$ of $A$, a binary relation $D$ on $A$, and a unary predicate $P$, and states that:

$$P[C]$$

provided the parameters meet the following conditions:

- $P[B]$,
- $D$ reduces $B$ to $C$, and
- For all elements $x, y$ of $A$ such that $D$ reduces $B$ to $x$ and $(x, y) \in D$ and $P[x]$ holds $P[y]$.

One can prove the following propositions:

(25) For every reduction sequence $t$ w.r.t. the construction reduction $X$ such that $s = t(1)$ holds $\text{rng } t \subseteq \text{coset } s$.

(26) $\text{coset } s = \{s_1 : \text{the construction reduction } X \text{ reduces } s \text{ to } s_1\}$.

Let us consider $X, s$. The functor core $s$ yields a stack of $X$ and is defined by the conditions (Def. 19).

(Def. 19)(i) empty(core $s$), and

(ii) there exists a the carrier’ of $X$-valued reduction sequence $t$ w.r.t. the construction reduction $X$ such that $t(1) = s$ and $t(\text{len } t) = \text{core } s$ and for every $i$ such that $1 \leq i < \text{len } t$ holds not empty($t_i$) and $t_i+1 = \text{pop}(t_i)$.

The following propositions are true:

(27) If empty($s$), then core $s = s$.

(28) core push($e, s$) = core $s$.

(29) If not empty($s$), then core pop $s = \text{core } s$.

(30) core $s \in \text{coset } s$.

(31) For every element $x$ of (the carrier of $X$)* there exists $s_1$ such that $|s_1| = x$ and $s_1 \in \text{coset } s$.

(32) If $s_1 \in \text{coset } s$, then core $s_1 = \text{core } s$.

(33) If $s_1, s_2 \in \text{coset } s$ and $|s_1| = |s_2|$, then $s_1 = s_2$.

(34) There exists $s$ such that coset $s_1 \cap [s_2]_{== X} = \{s\}$.

### 5. Quotient Stack System

Let us consider $X$. The functor $X_{/=}$ yields a strict stack system and is defined by the conditions (Def. 20).

(Def. 20)(i) The carrier of $X_{/=}$ = the carrier of $X$,

(ii) the carrier’ of $X_{/=}$ = Classes $== X$,

(iii) the empty stacks of $X_{/=}$ = \{the empty stacks of $X$\},

(iv) the push function of $X_{/=}$ = (the push function of $X$)$_{/= X}$,

(v) the pop function of $X_{/=}$ =

\[ ((\text{the pop function of } X) + \text{id}_{\text{the empty stacks of } X})_{/= X}, \] and
(vi) for every choice function $f$ of Classes $==X$ holds the top function of $X/==$ = (the top function of $X$) $\cdot f + \cdot (the$ empty stacks of $X$, the element of the carrier of $X$).

Let us consider $X$. One can verify that $X/==$ is non empty and non void.

The following propositions are true:

(35) For every stack $S$ of $X/==$ there exists $s$ such that $S = [s]=_{==X}$. 
(36) $[s]=_{==X}$ is a stack of $X/==$.
(37) For every stack $S$ of $X/==$ such that $S = [s]=_{==X}$ holds empty($s$) iff empty($S$).
(38) For every stack $S$ of $X/==$ holds empty($S$) iff $S$ = the empty stacks of $X$.
(39) For every stack $S$ of $X/==$ and for every element $E$ of $X/==$ such that $S = [s]=_{==X}$ and $E = e$ holds push($e,s$) $\in$ push($E,S$) and $[\text{push}(e,s)]=_{==X} = \text{push}(E,S)$.
(40) For every stack $S$ of $X/==$ such that $S = [s]=_{==X}$ and not empty($s$) holds pop $s \in$ pop $S$ and $[\text{pop}(s)]=_{==X} = \text{pop}(S)$.
(41) For every stack $S$ of $X/==$ such that $S = [s]=_{==X}$ and not empty($s$) holds top $S = \text{top}(s)$.

Let us consider $X$. One can verify the following observations:

- $X/==$ is pop-finite,
- $X/==$ is push-pop,
- $X/==$ is top-push,
- $X/==$ is pop-push, and
- $X/==$ is push-non-empty.

Next we state the proposition

(42) For every stack $S$ of $X/==$ such that $S = [s]=_{==X}$ holds $|S| = |s|.$

Let us consider $X$. Note that $X/==$ is proper for identity.

Let us note that there exists a stack algebra which is proper for identity.

6. REPRESENTATION THEOREM FOR STACKS

Let $X_1$, $X_2$ be stack algebras and let $F$, $G$ be functions. We say that $F$ and $G$ form isomorphism between $X_1$ and $X_2$ if and only if the conditions (Def. 21) are satisfied.

(Def. 21) $\text{dom } F = \text{the carrier of } X_1$ and $\text{rng } F = \text{the carrier of } X_2$ and $F$ is one-to-one and $\text{dom } G = \text{the carrier’ of } X_1$ and $\text{rng } G = \text{the carrier’ of } X_2$ and $G$ is one-to-one and for every stack $s_1$ of $X_1$ and for every stack $s_2$ of $X_2$ such that $s_2 = G(s_1)$ holds empty($s_1$) iff empty($s_2$) and if not empty($s_1$), then $\text{pop } s_2 = G(\text{pop } s_1)$ and $\text{top } s_2 = F(\text{top } s_1)$ and for every element
\( e_1 \) of \( X_1 \) and for every element \( e_2 \) of \( X_2 \) such that \( e_2 = F(e_1) \) holds \( \text{push}(e_2, s_2) = G(\text{push}(e_1, s_1)) \).

We use the following convention: \( X_1, X_2, X_3 \) are stack algebras and \( F, F_1, F_2, G, G_1, G_2 \) are functions.

The following propositions are true:

(43) \( \text{id} \) the carrier of \( X \) and \( \text{id} \) the carrier' of \( X \) form isomorphism between \( X \) and \( X \).

(44) If \( F \) and \( G \) form isomorphism between \( X_1 \) and \( X_2 \), then \( F^{-1} \) and \( G^{-1} \) form isomorphism between \( X_2 \) and \( X_1 \).

(45) Suppose \( F_1 \) and \( G_1 \) form isomorphism between \( X_1 \) and \( X_2 \) and \( F_2 \) and \( G_2 \) form isomorphism between \( X_2 \) and \( X_3 \). Then \( F_2 \cdot F_1 \) and \( G_2 \cdot G_1 \) form isomorphism between \( X_1 \) and \( X_3 \).

(46) Suppose \( F \) and \( G \) form isomorphism between \( X_1 \) and \( X_2 \). Let \( s_1 \) be a stack of \( X_1 \) and \( s_2 \) be a stack of \( X_2 \). If \( s_2 = G(s_1) \), then \( |s_2| = F \cdot |s_1| \).

Let \( X_1, X_2 \) be stack algebras. We say that \( X_1 \) and \( X_2 \) are isomorphic if and only if:

(Def. 22) There exist functions \( F, G \) such that \( F \) and \( G \) form isomorphism between \( X_1 \) and \( X_2 \).

Let us notice that the predicate \( X_1 \) and \( X_2 \) are isomorphic is reflexive and symmetric.

We now state four propositions:

(47) If \( X_1 \) and \( X_2 \) are isomorphic and \( X_2 \) and \( X_3 \) are isomorphic, then \( X_1 \) and \( X_3 \) are isomorphic.

(48) If \( X_1 \) and \( X_2 \) are isomorphic and \( X_1 \) is proper for identity, then \( X_2 \) is proper for identity.

(49) Let \( X \) be a proper for identity stack algebra. Then there exists \( G \) such that

(i) for every stack \( s \) of \( X \) holds \( G(s) = |s| \), and

(ii) \( \text{id} \) the carrier of \( X \) and \( G \) form isomorphism between \( X \) and standard stack system over the carrier of \( X \).

(50) Let \( X \) be a proper for identity stack algebra. Then \( X \) and standard stack system over the carrier of \( X \) are isomorphic.

REFERENCES


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