The Borsuk-Ulam Theorem

Artur Korniłowicz¹
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok
Poland

Marco Riccardi
Via del Pero 102
54038 Montignoso
Italy

Summary. The Borsuk-Ulam theorem about antipodals is proven, [18, pp. 32–33].

MML identifier: BORSUK_7, version: 7.12.02 4.176.1140

The notation and terminology used here have been introduced in the following papers: [33], [36], [15], [16], [2], [5], [28], [35], [13], [26], [20], [30], [4], [34], [6], [7], [8], [38], [27], [1], [3], [9], [29], [31], [19], [41], [42], [39], [11], [43], [37], [40], [25], [32], [14], [23], [24], [22], [12], [21], [17], and [10].

1. Preliminaries

For simplicity, we adopt the following rules: a, b, x, y, z, X, Y, Z denote sets, n denotes a natural number, i denotes an integer, r, r_1, r_2, r_3, s denote real numbers, c, c_1, c_2 denote complex numbers, and p denotes a point of $\mathbb{C}_T^n$.

Let us observe that every element of $\mathbb{Q}$ is irrational.

Next we state a number of propositions:

1. If $0 \leq r$ and $0 \leq s$ and $r^2 = s^2$, then $r = s$.
2. If $\text{frac}(r) \geq \text{frac}(s)$, then $\text{frac}(r - s) = \text{frac}(r) - \text{frac}(s)$.
3. If $\text{frac}(r) < \text{frac}(s)$, then $\text{frac}(r - s) = (\text{frac}(r) - \text{frac}(s)) + 1$.

¹This work has been supported by the Polish Ministry of Science and Higher Education project “Managing a Large Repository of Computer-verified Mathematical Knowledge” (N N519 385136).
(4) There exists $i$ such that $\frac{r - s}{\pi} = (\frac{r}{\pi} - \frac{s}{\pi}) + i$ but $i = 0$ or $i = 1$.

(5) If $\sin r = 0$, then $r = 2 \cdot \pi \cdot \frac{r}{2\pi}$ or $r = \pi + 2 \cdot \pi \cdot \frac{r}{2\pi}$.

(6) If $\cos r = 0$, then $r = \frac{r}{2} + 2 \cdot \pi \cdot \frac{r}{2\pi}$ or $r = \frac{3\pi}{2} + 2 \cdot \pi \cdot \frac{r}{2\pi}$.

(7) If $\sin r = 0$, then there exists $i$ such that $r = \pi \cdot i$.

(8) If $\cos r = 0$, then there exists $i$ such that $r = \frac{\pi}{2} + \pi \cdot i$.

(9) If $\sin r = \sin s$, then there exists $i$ such that $r = s + 2 \cdot \pi \cdot i$ or $r = (\pi - s) + 2 \cdot \pi \cdot i$.

(10) If $\cos r = \cos s$, then there exists $i$ such that $r = s + 2\pi \cdot i$ or $r = -s + 2\pi \cdot i$.

(11) If $\sin r = \sin s$ and $\cos r = \cos s$, then there exists $i$ such that $r = s + 2\pi \cdot i$.

(12) If $|c_1| = |c_2|$ and $\text{Arg} c_1 = \text{Arg} c_2 + 2 \cdot \pi \cdot i$, then $c_1 = c_2$.

Let $f$ be a one-to-one complex-valued function and let us consider $c$. One can verify that $f + c$ is one-to-one.

Let $f$ be a one-to-one complex-valued function and let us consider $c$. Note that $f - c$ is one-to-one.

One can prove the following propositions:

(13) For every complex-valued finite sequence $f$ holds $\text{len}(-f) = \text{len} f$.

(14) $-\left(\underbrace{0,\ldots,0}_{n}\right) = \left(\underbrace{0,\ldots,0}_{n}\right)$.

(15) For every complex-valued function $f$ such that $f \neq \left(\underbrace{0,\ldots,0}_{n}\right)$ holds $-f \neq \left(\underbrace{0,\ldots,0}_{n}\right)$.

(16) $\left\langle r_1, r_2, r_3 \right\rangle = \left\langle r_1^2, r_2^2, r_3^2 \right\rangle$.

(17) $\sum^2 \langle r_1, r_2, r_3 \rangle = r_1^2 + r_2^2 + r_3^2$.

(18) For every complex-valued finite sequence $f$ holds $(c \cdot f)^2 = c^2 \cdot f^2$.

(19) For every complex-valued finite sequence $f$ holds $(f/c)^2 = f^2/c^2$.

(20) For every real-valued finite sequence $f$ such that $\sum f \neq 0$ holds $\sum(f/\sum f) = 1$.

Let $a$, $b$, $c$, $x$, $y$, $z$ be sets. The functor $[a \mapsto x, b \mapsto y, c \mapsto z]$ is defined by:

(Def. 1) $[a \mapsto x, b \mapsto y, c \mapsto z] = [a \mapsto x, b \mapsto y] + (c \mapsto z)$.

Let $a$, $b$, $c$, $x$, $y$, $z$ be sets. One can check that $[a \mapsto x, b \mapsto y, c \mapsto z]$ is function-like and relation-like.

The following propositions are true:

(21) $\text{dom}([a \mapsto x, b \mapsto y, c \mapsto z]) = \{a, b, c\}$.

(22) $\text{rng}([a \mapsto x, b \mapsto y, c \mapsto z]) \subseteq \{x, y, z\}$.

(23) $[a \mapsto x, a \mapsto y, a \mapsto z] = [a \mapsto z]$.

(24) $[a \mapsto x, a \mapsto y, b \mapsto z] = [a \mapsto y, b \mapsto z]$.

(25) If $a \neq b$, then $[a \mapsto x, b \mapsto y, a \mapsto z] = [a \mapsto z, b \mapsto y]$.
(26) \([a \mapsto x, b \mapsto y, b \mapsto z] = [a \mapsto x, b \mapsto z]\).
(27) If \(a \neq b\) and \(a \neq c\), then \(([a \mapsto x, b \mapsto y, c \mapsto z])(a) = x\).
(28) If \(a, b, c\) are mutually different, then \(([a \mapsto x, b \mapsto y, c \mapsto z])(a) = x\) and \(([a \mapsto x, b \mapsto y, c \mapsto z])(b) = y\) and \(([a \mapsto x, b \mapsto y, c \mapsto z])(c) = z\).
(29) For every function \(f\) such that \(\text{dom } f = \{a, b, c\}\) and \(f(a) = x\) and \(f(b) = y\) and \(f(c) = z\), holds \(f = [a \mapsto x, b \mapsto y, c \mapsto z]\).
(30) \((a, b, c) = [1 \mapsto a, 2 \mapsto b, 3 \mapsto c]\).
(31) If \(a, b, c\) are mutually different, then \(\prod([a \mapsto \{x\}, b \mapsto \{y\}, c \mapsto \{z\}]) = \{a \mapsto x, b \mapsto y, c \mapsto z\}\).
(32) For all sets \(A, B, C, D, E, F\) such that \(A \subseteq B\) and \(C \subseteq D\) and \(E \subseteq F\) holds \(\prod([a \mapsto A, b \mapsto C, c \mapsto E]) \subseteq \prod([a \mapsto B, b \mapsto D, c \mapsto F])\).
(33) If \(a, b, c\) are mutually different and \(x \in X\) and \(y \in Y\) and \(z \in Z\), then \([a \mapsto x, b \mapsto y, c \mapsto z] \in \prod([a \mapsto X, b \mapsto Y, c \mapsto Z])\).

Let \(f\) be a function. We say that \(f\) is odd if and only if:

(**Def. 2**) For all complex-valued functions \(x, y\) such that \(x, -x \in \text{dom } f\) and \(y = f(x)\) holds \(f(-x) = -y\).

Let us mention that \(\emptyset\) is odd.

Let us observe that there exists a function which is odd and complex-functions-valued.

The following propositions are true:

(34) For every point \(p\) of \(\mathcal{E}^3\) holds \(\overline{p} = (p_1)^2, (p_2)^2, (p_3)^2\).
(35) For every point \(p\) of \(\mathcal{E}^3\) holds \(\sum p = (p_1)^2 + (p_2)^2 + (p_3)^2\).

The following two propositions are true:

(36) For every subset \(S\) of \(\mathbb{R}^1\) such that \(S = Q\) holds \(Q \cap [-\infty, r[\) is an open subset of \(\mathbb{R}^1\)\(\cup S\).
(37) For every subset \(S\) of \(\mathbb{R}^1\) such that \(S = Q\) holds \(Q \cap [r, +\infty[\) is an open subset of \(\mathbb{R}^1\)\(\cup S\).

Let \(X\) be a connected non empty topological space, let \(Y\) be a non empty topological space, and let \(f\) be a continuous function from \(X\) into \(Y\). Note that \(\text{Im } f\) is connected.

Next we state two propositions:

(38) Let \(S\) be a subset of \(\mathbb{R}^1\). Suppose \(S = Q\). Let \(T\) be a connected topological space and \(f\) be a function from \(T\) into \(\mathbb{R}^1\)\(\cup S\). If \(f\) is continuous, then \(f\) is constant.
(39) Let \(a, b\) be real numbers, \(f\) be a continuous function from \([a, b]_T\) into \(\mathbb{R}^1\), and \(g\) be a partial function from \(\mathbb{R}\) to \(\mathbb{R}\). If \(a \leq b\) and \(f = g\), then \(g\) is continuous.

Let \(s\) be a point of \(\mathbb{R}^1\) and let \(r\) be a real number. Then \(s + r\) is a point of \(\mathbb{R}^1\).
Let $s$ be a point of $\mathbb{R}^1$ and let $r$ be a real number. Then $s - r$ is a point of $\mathbb{R}^1$.

Let $X$ be a set, let $f$ be a function from $X$ into $\mathbb{R}^1$, and let us consider $r$. Then $f + r$ is a function from $X$ into $\mathbb{R}^1$.

Let $X$ be a set, let $f$ be a function from $X$ into $\mathbb{R}^1$, and let us consider $r$. Then $f - r$ is a function from $X$ into $\mathbb{R}^1$.

Let $s, t$ be points of $\mathbb{R}^1$, let $f$ be a path from $s$ to $t$, and let $r$ be a real number. Then $f + r$ is a path from $s + r$ to $t + r$. Then $f - r$ is a path from $s - r$ to $t - r$.

The point $c[100]$ of TopUnitCircle 3 is defined by:

(Def. 3) $c[100] = \{0\}$.

The point $c[-100]$ of TopUnitCircle 3 is defined by:

(Def. 4) $c[-100] = \{-1, 0, 0\}$.

Next we state several propositions:

(40) $-c[100] = c[-100]$.

(41) $-c[-100] = c[100]$.

(42) $c[100] - (-c[100]) = [2, 0, 0]$.

(43) For every point $p$ of $\mathcal{E}_3^1$ holds $|p| \cdot \cos \arg p$ and $p_2 = |p| \cdot \sin \arg p$.

(44) For every point $p$ of $\mathcal{E}_3^2$ holds $p = \text{cpx2euc}(|p| \cdot \cos \arg p + |p| \cdot \sin \arg p \cdot i)$.

(45) For all points $p_1, p_2$ of $\mathcal{E}_3^2$ such that $|p_1| = |p_2|$ and $\arg p_1 = \arg p_2 + 2 \cdot \pi \cdot i$ holds $p_1 = p_2$.

One can prove the following propositions:

(46) For every point $p$ of $\mathcal{E}_3^2$ such that $p = \text{CircleMap}(r)$ holds $\arg p = 2 \cdot \pi \cdot \text{frac} r$.

(47) Let $p_1, p_2$ be points of $\mathcal{E}_3^2$ and $u_1, u_2$ be points of $\mathcal{E}^3$. If $u_1 = p_1$ and $u_2 = p_2$, then $\rho^3(u_1, u_2) = \sqrt{((p_1)_1 - (p_2)_1)^2 + ((p_1)_2 - (p_2)_2)^2 + ((p_1)_3 - (p_2)_3)^2}$.

(48) Let $p$ be a point of $\mathcal{E}_3^1$ and $e$ be a point of $\mathcal{E}^3$. If $p = e$ and $p_3 = 0$, then $\Pi([1 \mapsto |p_1| - \frac{r}{\sqrt{2}}, p_1 + \frac{r}{\sqrt{2}}, 2 \mapsto |p_2| - \frac{r}{\sqrt{2}}, p_2 + \frac{r}{\sqrt{2}}, 3 \mapsto \{0\}) \subseteq \text{Ball}(e, r)$.

(49) For every real number $s$ holds $c \ominus s = c \ominus s + 2 \cdot \pi \cdot i$.

(50) For every real number $s$ holds $\text{Rotate } s = \text{Rotate}(s + 2 \cdot \pi \cdot i)$.

(51) For every real number $s$ and for every point $p$ of $\mathcal{E}_3^2$ holds $|(\text{Rotate } s)(p)| = |p|$.

(52) For every real number $s$ and for every point $p$ of $\mathcal{E}_3^2$ holds $\arg(\text{Rotate } s)(p) = \arg(\text{euc2cpx}(p) \ominus s)$.

(53) For every real number $s$ and for every point $p$ of $\mathcal{E}_3^2$ such that $p \neq 0_{\mathcal{E}_3^2}$ there exists $i$ such that $\arg(\text{Rotate } s)(p) = s + \arg p + 2 \cdot \pi \cdot i$.

(54) For every real number $s$ holds $(\text{Rotate } s)(0_{\mathcal{E}_3^2}) = 0_{\mathcal{E}_3^2}$.
(55) For every real number $s$ and for every point $p$ of $\mathcal{E}_T^2$ such that $(\text{Rotate } s)(p) = 0_{\mathcal{E}_T^2}$ holds $p = 0_{\mathcal{E}_T^2}$.

(56) For every real number $s$ and for every point $p$ of $\mathcal{E}_T^2$ holds $(\text{Rotate } s)((\text{Rotate } (-s))(p)) = p$.

(57) For every real number $s$ holds $\text{Rotate } s \cdot \text{Rotate } (-s) = \text{id}_{\mathcal{E}_T^2}$.

(58) For every real number $s$ and for every point $p$ of $\mathcal{E}_T^2$ holds $p \in \text{Sphere}((0_{\mathcal{E}_T^2}), r)$ iff $(\text{Rotate } s)(p) \in \text{Sphere}((0_{\mathcal{E}_T^2}), r)$.

(59) For every non negative real number $r$ and for every real number $s$ holds $(\text{Rotate } s)^{\circ} \text{Sphere}((0_{\mathcal{E}_T^2}), r) = \text{Sphere}((0_{\mathcal{E}_T^2}), r)$.

Let $r$ be a non negative real number and let $s$ be a real number. The functor $\text{RotateCircle}(r,s)$ yields a function from $T_{\text{circle}}(0_{\mathcal{E}_T^2},r)$ into $T_{\text{circle}}(0_{\mathcal{E}_T^2},r)$ and is defined by:

(Def. 5) $\text{RotateCircle}(r,s) = \text{Rotate } s \upharpoonright T_{\text{circle}}(0_{\mathcal{E}_T^2},r)$.

Let $r$ be a non negative real number and let $s$ be a real number. Note that $\text{RotateCircle}(r,s)$ is homeomorphism.

One can prove the following proposition

(60) For every point $p$ of $\mathcal{E}_T^2$ such that $p = \text{CircleMap}(r_2)$ holds $(\text{RotateCircle}(1, (\text{Arg } p))(\text{CircleMap}(r_1))) = \text{CircleMap}(r_1 - r_2)$.

2. ON THE ANTIPODALS

Let $n$ be a non empty natural number, let $p$ be a point of $\mathcal{E}_T^n$, and let $r$ be a non negative real number. The functor $\text{CircleIso}(p,r)$ yields a function from $\text{TopUnitCircle } n$ into $T_{\text{circle}}(p,r)$ and is defined as follows:

(Def. 6) For every point $a$ of $\text{TopUnitCircle } n$ and for every point $b$ of $\mathcal{E}_T^n$ such that $a = b$ holds $(\text{CircleIso}(p,r))(a) = r \cdot b + p$.

Let $n$ be a non empty natural number, let $p$ be a point of $\mathcal{E}_T^n$, and let $r$ be a positive real number. Note that $\text{CircleIso}(p,r)$ is homeomorphism.

The function $\text{SphereMap}$ from $\mathbb{R}^4$ into $\text{TopUnitCircle } 3$ is defined by:

(Def. 7) For every real number $x$ holds $(\text{SphereMap})(x) = [\cos(2 \cdot \pi \cdot x), \sin(2 \cdot \pi \cdot x), 0]$.

We now state the proposition

(61) $(\text{SphereMap})(i) = c[100]$.

Let us note that $\text{SphereMap}$ is continuous.

Let $r$ be a real number. The functor $\text{eLoop } r$ yields a function from $\mathbb{I}$ into $\text{TopUnitCircle } 3$ and is defined as follows:

(Def. 8) For every point $x$ of $\mathbb{I}$ holds $(\text{eLoop } r)(x) = [\cos(2 \cdot \pi \cdot r \cdot x), \sin(2 \cdot \pi \cdot r \cdot x), 0]$.

We now state the proposition

(62) $\text{eLoop } r = \text{SphereMap} \cdot \text{ExtendInt } r$. 
Let us consider $i$. Then eLoop $i$ is a loop of $c[100]$.

One can check that eLoop $i$ is null-homotopic as a loop of $c[100]$.

One can prove the following proposition

\[(\text{Def. 9})\] $(R^n \rightarrow S^1)p$ yields a point of $T\text{circle}(0_{\mathbb{E}^n}, 1)$ and is defined by:

\[(\text{Def. 10})\] For all points $x$ of $T\text{circle}(0_{\mathbb{E}^n}, 1)$ such that $y = -x$ holds

$$((S^{n+1} \rightarrow S^n)f)(x) = (R^n \rightarrow S^1)(f(x) - f(y)).$$

Let $x_0$, $y_0$ be points of $T\text{circle}2$, let $x_1$ be a set, and let $f$ be a path from $x_0$ to $y_0$. Let us assume that $x_1 \in \text{CircleMap}^{-1}(\{x_0\})$. The functor $\text{liftPath}(f, x_1)$ yielding a function from $I$ into $\mathbb{R}^1$ is defined by the conditions (Def. 11).

\[(\text{Def. 11}(i))\] $(\text{liftPath}(f, x_1))(0) = x_1$,

\[(\text{ii})\] $f = \text{CircleMap} \cdot \text{liftPath}(f, x_1)$,

\[(\text{iii})\] $\text{liftPath}(f, x_1)$ is continuous, and

\[(\text{iv})\] for every function $f_1$ from $I$ into $\mathbb{R}^1$ such that $f_1$ is continuous and $f = \text{CircleMap} \cdot f_1$ and $f_1(0) = x_1$ holds $\text{liftPath}(f, x_1) = f_1$.

Let $n$ be a natural number, let $p$, $x$, $y$ be points of $\mathbb{E}^n_T$, and let $r$ be a real number. We say that $x$ and $y$ are antipodals of $p$ and $r$ if and only if:

\[(\text{Def. 12})\] $x$ is a point of $T\text{circle}(p, r)$ and $y$ is a point of $T\text{circle}(p, r)$ and $p \in \mathcal{L}(x, y)$.

Let $n$ be a natural number, let $p$, $x$, $y$ be points of $\mathbb{E}^n_T$, let $r$ be a real number, and let $f$ be a function. We say that $x$ and $y$ are antipodals of $p$, $r$ and $f$ if and only if:

\[(\text{Def. 13})\] $x$ and $y$ are antipodals of $p$ and $r$ and $x, y \in \text{dom} f$ and $f(x) = f(y)$.

Let $m, n$ be natural numbers, let $p$ be a point of $\mathbb{E}^m_T$, let $r$ be a real number, and let $f$ be a function from $T\text{circle}(p, r)$ into $\mathbb{E}^n_T$. We say that $f$ has antipodals if and only if:

\[(\text{Def. 14})\] There exist points $x, y$ of $\mathbb{E}^m_T$ such that $x$ and $y$ are antipodals of $p$, $r$ and $f$.

Let $m, n$ be natural numbers, let $p$ be a point of $\mathbb{E}^m_T$, let $r$ be a real number, and let $f$ be a function from $T\text{circle}(p, r)$ into $\mathbb{E}^n_T$. We introduce $f$ is without antipodals as an antonym of $f$ has antipodals.

One can prove the following propositions:
(64) Let \( n \) be a non empty natural number, \( r \) be a non negative real number, and \( x \) be a point of \( \mathcal{E}_T^n \). Suppose \( x \) is a point of \( T\text{circle}(0_{\mathcal{E}_T^n}, r) \). Then \( x \) and \(-x\) are antipodals of \( 0_{\mathcal{E}_T^n} \) and \( r \).

(65) Let \( n \) be a non empty natural number, \( p, x, y, x_2, y_1 \) be points of \( \mathcal{E}_T^n \), and \( r \) be a positive real number. Suppose \( x \) and \( y \) are antipodals of \( 0_{\mathcal{E}_T^n} \) and \( 1 \) and \( x_2 = (\text{CircleIso}(p, r))(x) \) and \( y_1 = (\text{CircleIso}(p, r))(y) \). Then \( x_2 \) and \( y_1 \) are antipodals of \( p \) and \( r \).

(66) Let \( f \) be a function from \( T\text{circle}(0_{\mathcal{E}_T^{n+1}}, 1) \) into \( \mathcal{E}_T^n \) and \( x \) be a point of \( T\text{circle}(0_{\mathcal{E}_T^{n+1}}, 1) \). If \( f \) is without antipodals, then \( f(x) - f(-x) \neq 0_{\mathcal{E}_T^n} \).

(67) For every function \( f \) from \( T\text{circle}(0_{\mathcal{E}_T^{n+1}}, 1) \) into \( \mathcal{E}_T^n \) such that \( f \) is without antipodals holds \((S^{n+1} \rightarrow S^n) f \) is odd.

(68) Let \( f \) be a function from \( T\text{circle}(0_{\mathcal{E}_T^{n+1}}, 1) \) into \( \mathcal{E}_T^n \) and \( g, B_1 \) be functions from \( T\text{circle}(0_{\mathcal{E}_T^{n+1}}, 1) \) into \( \mathcal{E}_T^n \). If \( g = f \circ - \) and \( B_1 = f - g \) and \( f \) is without antipodals, then \((S^{n+1} \rightarrow S^n) f = B_1/(n \text{Norm} F \cdot B_1) \).

Let us consider \( n \), let \( r \) be a negative real number, and let \( p \) be a point of \( \mathcal{E}_T^{n+1} \). Observe that every function from \( T\text{circle}(p, r) \) into \( \mathcal{E}_T^n \) is without antipodals.

Let \( r \) be a non negative real number and let \( p \) be a point of \( \mathcal{E}_T^3 \). Note that every function from \( T\text{circle}(p, r) \) into \( \mathcal{E}_T^2 \) which is continuous also has antipodals.\(^2\)

**References**


\(^2\)The Borsuk-Ulam Theorem

Received September 20, 2011