The Gödel Completeness Theorem for Uncountable Languages\footnote{This article is part of the first author’s Bachelor thesis under the supervision of the second author.}

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Summary. This article is the second in a series of two Mizar articles constituting a formal proof of the Gödel Completeness theorem [15] for uncountably large languages. We follow the proof given in [16]. The present article contains the techniques required to expand a theory such that the expanded theory contains witnesses and is negation faithful. Then the completeness theorem follows immediately.

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The notation and terminology used here have been introduced in the following papers: [8], [1], [3], [10], [19], [5], [14], [11], [12], [7], [6], [22], [2], [4], [17], [18], [23], [20], [9], [21], and [13].
1. Formula-Constant Extension

For simplicity, we use the following convention: $A_1$ denotes an alphabet, $P_1$ denotes a consistent subset of CQC-WFF $A_1$, $P_2$ denotes a subset of CQC-WFF $A_1$, $p, q, r, s$ denote elements of CQC-WFF $A_1$, $A$ denotes a nonempty set, $J$ denotes an interpretation of $A_1$ and $A$, $v$ denotes an element of the valuations in $A_1$ and $A$, $n, k$ denote elements of $\mathbb{N}$, $x$ denotes a bound variable of $A_1$, and $A_2$ denotes an $A_1$-expanding alphabet.

Let us consider $A_1$ and let $P_1$ be a subset of CQC-WFF $A_1$. We say that $P_1$ is satisfiable if and only if:

(Def. 1) There exist $A, J, v$ such that $J \models v P_1$.

In the sequel $J_2$ is an interpretation of $A_2$ and $A$ and $J_1$ is an interpretation of $A_1$ and $A$.

One can prove the following proposition

(1) There exists a set $s$ such that for all $p, x$ holds $\langle s, \langle x, p \rangle \rangle \notin \text{Symb} A_1$.

Let us consider $A_1$. A set is called a free symbol of $A_1$ if:

(Def. 2) For all $p, x$ holds $\langle s, \langle x, p \rangle \rangle \notin \text{Symb} A_1$.

Let us consider $A_1$. The functor $\text{FCEx} A_1$ yielding an $A_1$-expanding alphabet is defined as follows:

(Def. 3) $\text{FCEx} A_1 = \mathbb{N} \times (\text{Symb} A_1 \cup \{ \text{the free symbol of } A_1, \langle x, p \rangle \})$.

Let us consider $A_1, p, x$. The example of $p$ and $x$ yielding a bound variable of $\text{FCEx} A_1$ is defined as follows:

(Def. 4) The example of $p$ and $x = \langle 4, \{ \text{the free symbol of } A_1, \langle x, p \rangle \} \rangle$.

Let us consider $A_1, p, x$. The example formula of $p$ and $x$ yielding an element of CQC-WFF $\text{FCEx} A_1$ is defined by:

(Def. 5) The example formula of $p$ and $x = \neg F(\text{FCEx} A_1 - \text{Cast} x) (\text{FCEx} A_1 - \text{Cast} p) \lor (\text{FCEx} A_1 - \text{Cast} p) (\text{FCEx} A_1 - \text{Cast} x, \text{the example of } p \text{ and } x)$.

Let us consider $A_1$. The example formulae of $A_1$ yields a subset of CQC-WFF $\text{FCEx} A_1$ and is defined as follows:

(Def. 6) The example formulae of $A_1 = \{ \text{the example formula of } p \text{ and } x \}$.

One can prove the following proposition

(2) Let $k$ be an element of $\mathbb{N}$. Suppose $k > 0$. Then there exists a $k$-element finite sequence $F$ such that

(i) for every natural number $n$ such that $n \leq k$ and $1 \leq n$ holds $F(n)$ is an alphabet,

(ii) $F(1) = A_1$, and

(iii) for every natural number $n$ such that $n < k$ and $1 \leq n$ there exists an alphabet $A_2$ such that $F(n) = A_2$ and $F(n + 1) = \text{FCEx} A_2$. 
Let us consider $A_1$ and let $k$ be a natural number. A $k + 1$-element finite sequence is said to be a FCEx-sequence of $A_1$ and $k$ if it satisfies the conditions (Def. 7).

(Def. 7)(i) For every natural number $n$ such that $n \leq k + 1$ and $1 \leq n$ holds $it(n)$ is an alphabet,

(ii) $it(1) = A_1$, and

(iii) for every natural number $n$ such that $n < k + 1$ and $1 \leq n$ there exists an alphabet $A_2$ such that $it(n) = A_2$ and $it(n + 1) = \text{FCEx } A_2$.

The following propositions are true:

(3) For every natural number $k$ and for every FCEx-sequence $S$ of $A_1$ and $k$ holds $S(k + 1)$ is an alphabet.

(4) For every natural number $k$ and for every FCEx-sequence $S$ of $A_1$ and $k$ holds $S(k + 1)$ is an $A_1$-expanding alphabet.

Let us consider $A_1$ and let $k$ be a natural number. The $k$-th FCEx of $A_1$ yielding an $A_1$-expanding alphabet is defined as follows:

(Def. 8) The $k$-th FCEx of $A_1 = \text{the FCEx-sequence of } A_1 \text{ and } k(k + 1)$.

Let us consider $A_1$, $P_1$. A function is called an EF-sequence of $A_1$ and $P_1$ if it satisfies the conditions (Def. 9).

(Def. 9)(i) $\text{dom } it = \mathbb{N}$,

(ii) $it(0) = P_1$, and

(iii) for every natural number $n$ holds $it(n + 1) = it(n) \cup \text{the example formulae of the } n\text{-th FCEx of } A_1$.

Next we state two propositions:

(5) For every natural number $k$ holds $\text{FCEx(} \text{the } k\text{-th FCEx of } A_1) = \text{the } (k + 1)\text{-th FCEx of } A_1$.

(6) For all $k$, $n$ such that $n \leq k$ holds the $n$-th FCEx of $A_1 \subseteq \text{the } k\text{-th FCEx of } A_1$.

Let us consider $A_1$, $P_1$ and let $k$ be a natural number. The $k$-th EF of $A_1$ and $P_1$ yields a subset of CQC-WFF (the $k$-th FCEx of $A_1$) and is defined as follows:

(Def. 10) The $k$-th EF of $A_1$ and $P_1 = \text{the EF-sequence of } A_1 \text{ and } P_1(k)$.

One can prove the following propositions:

(7) For all $r$, $s$, $x$ holds $A_2\text{-Cast}(r \lor s) = A_2\text{-Cast } r \lor A_2\text{-Cast } s$ and $A_2\text{-Cast } \exists x r = \exists A_2\text{-Cast } x (A_2\text{-Cast } r)$.

(8) For all $p$, $q$, $A$, $J$, $v$ holds $J \models_v p \text{ or } J \models_v q$ iff $J \models_v p \lor q$.

(9) $P_1 \cup \text{the example formulae of } A_1$ is a consistent subset of CQC-WFF FCEx $A_1$. 
2. The Completeness Theorem

We now state four propositions:

(10) There exists an $A_1$-expanding alphabet $A_2$ and there exists a consistent subset $P_2$ of CQC-WFF $A_2$ such that $P_1 \subseteq P_2$ and $P_2$ has examples.

(11) $P_1 \cup \{p\}$ is consistent or $P_1 \cup \{\neg p\}$ is consistent.

(12) Let $P_2$ be a consistent subset of CQC-WFF $A_1$. Then there exists a consistent subset $T_1$ of CQC-WFF $A_1$ such that $T_1$ is negation faithful and $P_2 \subseteq T_1$.

(13) For every consistent subset $T_1$ of CQC-WFF $A_1$ such that $P_1 \subseteq T_1$ and $P_1$ has examples holds $T_1$ has examples.

Let us consider $A_1$. One can check that every subset of CQC-WFF $A_1$ which is consistent is also satisfiable.

We now state the proposition

(14)² If $P_2 \models p$, then $P_2 \vdash p$.

REFERENCES


²Completeness Theorem.


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