On $L^1$ Space Formed by Complex-Valued Partial Functions

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Summary. In this article, we formalized $L^1$ space formed by complex-valued partial functions [11], [15]. The real-valued case was formalized in [22] and this article is its generalization.

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The notation and terminology used here have been introduced in the following papers: [4], [10], [5], [19], [17], [6], [7], [1], [22], [3], [18], [13], [16], [8], [14], [23], [24], [12], [20], [21], [2], and [9].

1. Preliminaries of Complex Linear Space

Let $D$ be a non empty set and let $E$ be a complex-membered set. One can verify that every element of $D \rightarrow E$ is complex-valued.

Let $D$ be a non empty set, let $E$ be a complex-membered set, and let $F_1$, $F_2$ be elements of $D \rightarrow E$. Then $F_1 + F_2$ is an element of $D \rightarrow \mathbb{C}$. Then $F_1 - F_2$ is an element of $D \rightarrow \mathbb{C}$. Then $F_1 \cdot F_2$ is an element of $D \rightarrow \mathbb{C}$. Then $F_1/F_2$ is an element of $D \rightarrow \mathbb{C}$.

Let $D$ be a non empty set, let $E$ be a complex-membered set, let $F$ be an element of $D \rightarrow E$, and let $a$ be a complex number. Then $a \cdot F$ is an element of $D \rightarrow \mathbb{C}$.

Let $V$ be a non empty CLS structure and let $V_1$ be a subset of $V$. We say that $V_1$ is multiplicatively closed if and only if:
(Def. 1) For every complex number $a$ and for every vector $v$ of $V$ such that $v \in V_1$ holds $a \cdot v \in V_1$.

Next we state the proposition

(1) Let $V$ be a complex linear space and $V_1$ be a subset of $V$. Then $V_1$ is linearly closed if and only if $V_1$ is add closed and multiplicatively closed.

Let $V$ be a non empty CLS structure. One can verify that there exists a non empty subset of $V$ which is add closed and multiplicatively closed.

Let $X$ be a non empty CLS structure and let $X_1$ be a multiplicatively closed non empty subset of $X$. The functor $\cdot_{(X_1)}$ yields a function from $\mathbb{C} \times X_1$ into $X_1$ and is defined by:

(Def. 2) $\cdot_{(X_1)} = (\text{the external multiplication of } X) \downharpoonright (\mathbb{C} \times X_1)$.

In the sequel $a$, $b$, $r$ denote complex numbers and $V$ denotes a complex linear space.

We now state two propositions:

(2) Let $V$ be an Abelian add-associative right zeroed vector distributive scalar distributive scalar associative scalar unital non empty CLS structure, $V_1$ be a non empty subset of $V$, $d_1$ be an element of $V_1$, $A$ be a binary operation on $V_1$, and $M$ be a function from $\mathbb{C} \times V_1$ into $V_1$. Suppose $d_1 = 0_V$ and $A = (\text{the addition of } V) \downharpoonright (V_1)$ and $M = (\text{the external multiplication of } V) \downharpoonright (\mathbb{C} \times V_1)$. Then $\langle V_1, d_1, A, M \rangle$ is Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

(3) Let $V$ be an Abelian add-associative right zeroed vector distributive scalar distributive scalar associative scalar unital non empty CLS structure and $V_1$ be an add closed multiplicatively closed non empty subset of $V$. Suppose $0_V \in V_1$. Then $\langle V_1, 0_V, (\in V_1), \text{add } \downharpoonright (V_1, V), \cdot_{(V_1)} \rangle$ is Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

2. Quasi-Complex Linear Space of Partial Functions

We follow the rules: $A$, $B$ are non empty sets and $f$, $g$, $h$ are elements of $A \rightarrow \mathbb{C}$.

Let us consider $A$. The functor multcpfunc $A$ yielding a binary operation on $A \rightarrow \mathbb{C}$ is defined as follows:

(Def. 3) For all elements $f$, $g$ of $A \rightarrow \mathbb{C}$ holds $(\text{multcpfunc } A)(f, g) = f \cdot g$.

Let us consider $A$. The functor multcomplexcpfunc $A$ yielding a function from $\mathbb{C} \times (A \rightarrow \mathbb{C})$ into $A \rightarrow \mathbb{C}$ is defined by:

(Def. 4) For every complex number $a$ and for every element $f$ of $A \rightarrow \mathbb{C}$ holds $(\text{multcomplexcpfunc } A)(a, f) = a \cdot f$. 


Let $D$ be a non empty set. The functor addcpfunc $D$ yields a binary operation on $D \rightarrow \mathbb{C}$ and is defined as follows:

(Def. 5) For all elements $F_1, F_2$ of $D \rightarrow \mathbb{C}$ holds \( \text{addcpfunc}(D)(F_1, F_2) = F_1 + F_2 \).

Let $A$ be a set. The functor CPFuncZero $A$ yields an element of $A \rightarrow \mathbb{C}$ and is defined by:

(Def. 6) \( \text{CPFuncZero } A = A \mapsto 0 \mathbb{C} \).

Let $A$ be a set. The functor CPFuncUnit $A$ yielding an element of $A \rightarrow \mathbb{C}$ is defined as follows:

(Def. 7) \( \text{CPFuncUnit } A = A \mapsto 1 \mathbb{C} \).

The following propositions are true:

(4) \( h = (\text{addcpfunc } A)(f, g) \) iff \( \text{dom } h = \text{dom } f \cap \text{dom } g \) and for every element $x$ of $A$ such that $x \in \text{dom } h$ holds $h(x) = f(x) + g(x)$.

(5) \( h = (\text{multcpfunc } A)(f, g) \) iff \( \text{dom } h = \text{dom } f \cap \text{dom } g \) and for every element $x$ of $A$ such that $x \in \text{dom } h$ holds $h(x) = f(x) \cdot g(x)$.

(6) \( \text{CPFuncZero } A \neq \text{CPFuncUnit } A \).

(7) \( h = (\text{multcomplexcpfunc } A)(a, f) \) iff \( \text{dom } h = \text{dom } f \) and for every element $x$ of $A$ such that $x \in \text{dom } f$ holds $h(x) = a \cdot f(x)$.

Let us consider $A$. Note that addcpfunc $A$ is commutative and associative. Observe that multcpfunc $A$ is commutative and associative.

One can prove the following propositions:

(8) \( \text{CPFuncUnit } A \) is a unity w.r.t. \text{multcpfunc } A.

(9) \( \text{CPFuncZero } A \) is a unity w.r.t. \text{addcpfunc } A.

(10) \( (\text{addcpfunc } A)(f, (\text{multcomplexcpfunc } A)(-1 \mathbb{C}, f)) = \text{CPFuncZero } A \upharpoonright \text{dom } f \).

(11) \( \text{multcomplexcpfunc } A)(1 \mathbb{C}, f) = f \).

(12) \( \text{multcomplexcpfunc } A)(a, (\text{multcomplexcpfunc } A)(b, f)) = (\text{multcomplexcpfunc } A)(a \cdot b, f) \).

(13) \( (\text{addcpfunc } A)((\text{multcomplexcpfunc } A)(a, f), \text{multcomplexcpfunc } A)(b, f)) = (\text{multcomplexcpfunc } A)(a + b, f) \).

(14) \( \text{multcpfunc } A)(f, (\text{addcpfunc } A)(g, h)) = (\text{addcpfunc } A)((\text{multcpfunc } A)(f, g), (\text{multcpfunc } A)(f, h)) \).

(15) \( \text{multcpfunc } A)((\text{multcomplexcpfunc } A)(a, f), g) = (\text{multcomplexcpfunc } A)(a, (\text{multcpfunc } A)(f, g)) \).

Let us consider $A$. The functor CLSp PFunct $A$ yields a non empty CLS structure and is defined as follows:

(Def. 8) \( \text{CLSp PFunct } A = (A \rightarrow \mathbb{C}, \text{CPFuncZero } A, \text{addcpfunc } A, \text{multcomplexcpfunc } A) \).

In the sequel $u, v, w$ are vectors of CLSp PFunct $A$. 
Note that CLSp PFunct $A$ is strict, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

3. Quasi-Complex Linear Space of Integrable Functions

For simplicity, we use the following convention: $X$ is a non empty set, $x$ is an element of $X$, $S$ is a $\sigma$-field of subsets of $X$, $M$ is a $\sigma$-measure on $S$, $E$, $A$ are elements of $S$, and $f$, $g$, $h$, $f_1$, $g_1$ are partial functions from $X$ to $\mathbb{C}$.

Let us consider $X$ and let $f$ be a partial function from $X$ to $\mathbb{C}$. Note that $|f|$ is non-negative.

Next we state the proposition

(16) Let $f$ be a partial function from $X$ to $\mathbb{C}$. Suppose $\text{dom } f \in S$ and for every $x$ such that $x \in \text{dom } f$ holds $0 = f(x)$. Then $f$ is integrable on $M$ and $\int f \, dM = 0$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The functor $L_1 \text{CFuncts } M$ yielding a non empty subset of CLSp PFunct $X$ is defined by the condition (Def. 9).

(Def. 9) $L_1 \text{CFuncts } M = \{f; f \text{ ranges over partial functions from } X \text{ to } \mathbb{C}: \forall N_1: \text{element of } S (M(N_1) = 0 \land \text{dom } f = N_1^c \land f \text{ is integrable on } M)\}$.

The following propositions are true:

(17) If $f$, $g \in L_1 \text{CFuncts } M$, then $f + g \in L_1 \text{CFuncts } M$.
(18) If $f \in L_1 \text{CFuncts } M$, then $a \cdot f \in L_1 \text{CFuncts } M$.

Note that $L_1 \text{CFuncts } M$ is multiplicatively closed and add closed.

The functor CLSp $L_1 \text{Funct } M$ yielding a non empty CLS structure is defined by:

(Def. 10) $\text{CLSp } L_1 \text{Funct } M = \langle L_1 \text{CFuncts } M, 0_{\text{CLSp PFunct } X}(\in L_1 \text{CFuncts } M), \text{add } [(L_1 \text{CFuncts } M, \text{CLSp PFunct } X), L_1 \text{CFuncts } M] \rangle$.

One can verify that CLSp $L_1 \text{Funct } M$ is strict, Abelian, add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

4. Quotient Space of Quasi-Complex Linear Space of Integrable Functions

In the sequel $v$, $u$ are vectors of CLSp $L_1 \text{Funct } M$.

Next we state two propositions:

(19) If $f = v$ and $g = u$, then $f + g = v + u$.
(20) If $f = u$, then $a \cdot f = a \cdot u$. 
Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $f, g$ be partial functions from $X$ to $\mathbb{C}$. We say that $f$ a.e.cpfunc = $g$ and $M$ if and only if:

(Def. 11) There exists an element $E$ of $S$ such that $M(E) = 0$ and $f|E^c = g|E^c$.

We now state several propositions:

(21) Suppose $f = u$. Then

(i) $u + (-1_C) \cdot u = (X \mapsto 0_C)|\text{dom } f$, and

(ii) there exist partial functions $v, g$ from $X$ to $\mathbb{C}$ such that $v, g \in L_1\text{CFunct } M$ and $v = u + (-1_C) \cdot u$ and $g = X \mapsto 0_C$ and $v$ a.e.cpfunc = $g$ and $M$.

(22) $f$ a.e.cpfunc = $f$ and $M$.

(23) If $f$ a.e.cpfunc = $g$ and $M$, then $g$ a.e.cpfunc = $f$ and $M$.

(24) If $f$ a.e.cpfunc = $g$ and $M$ and $g$ a.e.cpfunc = $h$ and $M$, then $f$ a.e.cpfunc = $h$ and $M$.

(25) If $f$ a.e.cpfunc = $f_1$ and $M$ and $g$ a.e.cpfunc = $g_1$ and $M$, then $f + g$ a.e.cpfunc = $f_1 + g_1$ and $M$.

(26) If $f$ a.e.cpfunc = $g$ and $M$, then $a \cdot f$ a.e.cpfunc = $a \cdot g$ and $M$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The almost zero cfunctions of $M$ yields a non empty subset of $\text{CLSp } L_1\text{Funct } M$ and is defined by the condition (Def. 12).

(Def. 12) The almost zero cfunctions of $M = \{f; f$ ranges over partial functions from $X$ to $\mathbb{C}; f \in L_1\text{CFunct } M \land f$ a.e.cpfunc $= X \mapsto 0_C \text{ and } M\}$.

One can prove the following proposition

(27) $(X \mapsto 0_C) + (X \mapsto 0_C) = X \mapsto 0_C \text{ and } a \cdot (X \mapsto 0_C) = X \mapsto 0_C$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. One can check that the almost zero cfunctions of $M$ is add closed and multiplicatively closed.

One can prove the following proposition

(28) $0_{\text{CLSp } L_1\text{Funct } M} = X \mapsto 0_C \text{ and } 0_{\text{CLSp } L_1\text{Funct } M} \in$ the almost zero cfunctions of $M$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The clsp almost zero functions of $M$ yields a non empty CLS structure and is defined by the condition (Def. 13).

(Def. 13) The clsp almost zero functions of $M = \langle$ the almost zero cfunctions of $M, 0_{\text{CLSp } L_1\text{Funct } M}(\in$ the almost zero cfunctions of $M), \text{add}|(the$ almost zero cfunctions of $M, \text{CLSp } L_1\text{Funct } M), \langle$the almost zero cfunctions of $M\rangle\rangle$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. One can check that $\text{CLSp } L_1\text{Funct } M$ is strict, Abelian,
add-associative, right zeroed, vector distributive, scalar distributive, scalar associative, and scalar unital.

In the sequel $v, u$ are vectors of the clsp almost zero functions of $M$.

One can prove the following proposition

(29) If $f = v$ and $g = u$, then $f + g = v + u$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, let $M$ be a $\sigma$-measure on $S$, and let $f$ be a partial function from $X$ to $\mathbb{C}$. The functor $a.e$-Ceq-class$(f, M)$ yields a subset of $L_1$CFunct $M$ and is defined as follows:

(Def. 14) $a.e$-Ceq-class$(f, M) = \{g; g$ ranges over partial functions from $X$ to $\mathbb{C}$: $g \in L_1$CFunct $M \land f \in L_1$CFunct $M \land f \ a.e.cpfunc = g$ and $M\}$.

Next we state several propositions:

(30) If $f, g \in L_1$CFunct $M$, then $g \ a.e.cpfunc = f$ and $M$ iff $g \in a.e$-Ceq-class$(f, M)$.

(31) If $f \in L_1$CFunct $M$, then $f \in a.e$-Ceq-class$(f, M)$.

(32) If $f, g \in L_1$CFunct $M$, then $a.e$-Ceq-class$(f, M) = a.e$-Ceq-class$(g, M)$ iff $f \ a.e.cpfunc = g$ and $M$.

(33) If $f, g \in L_1$CFunct $M$, then $a.e$-Ceq-class$(f, M) = a.e$-Ceq-class$(g, M)$ iff $g \in a.e$-Ceq-class$(f, M)$.

(34) If $f, f_1, g, g_1 \in L_1$CFunct $M$ and $a.e$-Ceq-class$(f, M) = a.e$-Ceq-class$(f_1, M)$ and $a.e$-Ceq-class$(g, M) = a.e$-Ceq-class$(g_1, M)$, then $a.e$-Ceq-class$(f + g, M) = a.e$-Ceq-class$(f_1 + g_1, M)$.

(35) If $f, g \in L_1$CFunct $M$ and $a.e$-Ceq-class$(f, M) = a.e$-Ceq-class$(g, M)$, then $a.e$-Ceq-class$(a \cdot f, M) = a.e$-Ceq-class$(a \cdot g, M)$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The functor $CCosetSet M$ yields a non empty family of subsets of $L_1$CFunct $M$ and is defined by:

(Def. 15) $CCosetSet M = \{a.e$-Ceq-class$(f, M); f$ ranges over partial functions from $X$ to $\mathbb{C}$: $f \in L_1$CFunct $M\}$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The functor $addCCoset M$ yields a binary operation on $CCosetSet M$ and is defined by the condition (Def. 16).

(Def. 16) Let $A, B$ be elements of $CCosetSet M$ and $a, b$ be partial functions from $X$ to $\mathbb{C}$. If $a \in A$ and $b \in B$, then $(addCCoset M)(A, B) = a.e$-Ceq-class$(a + b, M)$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The functor $zeroCCoset M$ yielding an element of $CCosetSet M$ is defined by:

(Def. 17) $zeroCCoset M = a.e$-Ceq-class$(X \mapsto 0_\mathbb{C}, M)$.
Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The functor $\text{lmultCCoset} M$ yields a function from $\mathbb{C} \times \text{CCosetSet} M$ into $\text{CCosetSet} M$ and is defined by the condition (Def. 18).

(Def. 18) Let $z$ be a complex number, $A$ be an element of $\text{CCosetSet} M$, and $f$ be a partial function from $X$ to $\mathbb{C}$. If $f \in A$, then $(\text{lmultCCoset} M)(z, A) = a \cdot e^{-\text{Ceq-class}(z \cdot f, M)}$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The functor $\text{Pre-L-CSpace} M$ yields a strict Abelian add-associative right zeroed right complementable vector distributive scalar associative scalar unital non empty CLS structure and is defined by the conditions (Def. 19).

(Def. 19)(i) The carrier of $\text{Pre-L-CSpace} M = \text{CCosetSet} M$,
(ii) the addition of $\text{Pre-L-CSpace} M = \text{addCCoset} M$,
(iii) $0_{\text{Pre-L-CSpace} M} = \text{zeroCCoset} M$, and
(iv) the external multiplication of $\text{Pre-L-CSpace} M = \text{lmultCCoset} M$.

5. Complex Normed Space of Integrable Functions

Next we state several propositions:

(36) If $f, g \in L^1 \text{CFunctions} M$ and $f$ a.e.$\text{cpfunc} = g$ and $M$, then $\int f \, dM = \int g \, dM$.

(37) If $f$ is integrable on $M$, then $\int f \, dM \in \mathbb{C}$ and $\int |f| \, dM \in \mathbb{R}$ and $|f|$ is integrable on $M$.

(38) If $f, g \in L_1 \text{CFunctions} M$ and $f$ a.e.$\text{cpfunc} = g$ and $M$, then $|f| = M^a.e. \cdot |g|$ and $\int |f| \, dM = \int |g| \, dM$.

(39) If there exists a vector $x$ of $\text{Pre-L-CSpace} M$ such that $f, g \in x$, then $f$ a.e.$\text{cpfunc} = g$ and $M$ and $f, g \in L_1 \text{CFunctions} M$.

(40) There exists a function $N_2$ from the carrier of $\text{Pre-L-CSpace} M$ into $\mathbb{R}$ such that for every point $x$ of $\text{Pre-L-CSpace} M$ holds there exists a partial function $f$ from $X$ to $\mathbb{C}$ such that $f \in x$ and $N_2(x) = \int |f| \, dM$.

In the sequel $x$ is a point of $\text{Pre-L-CSpace} M$.

The following two propositions are true:

(41) If $f \in x$, then $f$ is integrable on $M$ and $f \in L_1 \text{CFunctions} M$ and $|f|$ is integrable on $M$.

(42) If $f, g \in x$, then $f$ a.e.$\text{cpfunc} = g$ and $M$ and $\int f \, dM = \int g \, dM$ and $\int |f| \, dM = \int |g| \, dM$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The functor $L^1 \text{-CNorm} M$ yields a function from the carrier of $\text{Pre-L-CSpace} M$ into $\mathbb{R}$ and is defined by:
(Def. 20) For every point $x$ of $\text{Pre-L-CSpace } M$ there exists a partial function $f$ from $X$ to $\mathbb{C}$ such that $f \in x$ and $(L-1-C\text{Norm } M)(x) = \int |f| \, dM$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. The functor $L-1\text{-CSpace } M$ yields a non empty complex normed space structure and is defined as follows:

(Def. 21) $L-1\text{-CSpace } M = \langle \text{the carrier of } \text{Pre-L-CSpace } M, \text{the zero of } \text{Pre-L-CSpace } M, \text{the addition of } \text{Pre-L-CSpace } M, \text{the external multiplication of } \text{Pre-L-CSpace } M, L-1\text{-CNorm } M \rangle$.

In the sequel $x$ denotes a point of $L-1\text{-CSpace } M$.

Next we state several propositions:

(43)(i) There exists a partial function $f$ from $X$ to $\mathbb{C}$ such that $f \in L^1_{\text{CFunctions }} M$ and $x = a\cdot \text{e-Ceq-class}(f, M)$ and $\|x\| = \int |f| \, dM$, and

(ii) for every partial function $f$ from $X$ to $\mathbb{C}$ such that $f \in x$ holds $\int |f| \, dM = \|x\|$.

(44) If $f \in x$, then $x = a\cdot \text{e-Ceq-class}(f, M)$ and $\|x\| = \int |f| \, dM$.

(45) If $f \in x$ and $g \in y$, then $f + g \in x + y$ and if $f \in x$, then $a \cdot f \in a \cdot x$.

(46) If $f \in L^1_{\text{CFunctions }} M$ and $\int |f| \, dM = 0$, then $f$ a.e.$\text{cpfunc} = X \mapsto 0_{\mathbb{C}}$ and $M$.

(47) If $f, g \in L^1_{\text{CFunctions }} M$, then $\int |f + g| \, dM \leq \int |f| \, dM + \int |g| \, dM$.

Let $X$ be a non empty set, let $S$ be a $\sigma$-field of subsets of $X$, and let $M$ be a $\sigma$-measure on $S$. One can check that $L-1\text{-CSpace } M$ is complex normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, and right complementable.

REFERENCES


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